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On the Diophantine Equations $p^4 + q^3 = z^2$ and $p^4 - q^3 = z^2$ when p, q are Distinct Odd Primes

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Abstract. In this paper we consider the two equations $p^4 + q^3 = z^2$ and $p^4 - q^3 = z^2$ in which p, q are distinct odd primes, and z is a positive integer. We establish that both equations have no solutions.

Keywords: Diophantine equations

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds. Among them are for example [1, 3, 4].

In this paper, we consider the two equations

$$p^4 + q^3 = z^2$$
$$p^4 - q^3 = z^2$$

in which p, q are distinct odd primes, and z is a positive integer.

2. The solutions of $p^4 + q^3 = z^2$ and $p^4 - q^3 = z^2$

In Theorem 2.1, we establish that the equations have no solutions.

Theorem 2.1. Suppose that p, q are distinct odd primes, and z is a positive integer. Then, the equations

(a) $p^4 + q^3 = z^2$,
(b) $p^4 - q^3 = z^2$

have no solutions.

Proof: **(a)** Suppose

$$p^4 + q^3 = z^2. \quad (1)$$

We shall assume that there exist distinct odd primes p, q, such that (1) has a solution, and reach a contradiction.

Equation (1) yields

$$q^3 = z^2 - p^4 = z^2 - (p^2)^2 = (z - p^2)(z + p^2).$$

Denote

$$z - p^2 = q^A, \qquad z + p^2 = q^B, \qquad A < B, \qquad A + B = 3,$$

where A, B are non-negative integers. Then $q^B - q^A$ yields

$$2p^2 = q^A(q^{B-A} - 1). \quad (2)$$

Since p, q are distinct primes, it follows from (2) that $A > 0$ is impossible, and hence $A = 0$. When $A = 0$ then $B = 3$, and (2) implies $2p^2 = q^3 - 1 = q^3 - 1^3$ or

$$2p^2 = (q - 1)(q^2 + q + 1). \quad (3)$$

When $q = 3$ in (3), then $2p^2 = 2 \cdot 13$ which is impossible. Therefore $q \neq 3$, and by our assumption $q > 3$. Since $2 \nmid (q^2 + q + 1)$ and $q > 3$, it follows that $2 \mid (q - 1)$. Denote $2M = q - 1$ where $M > 1$ is an integer. Thus, $q = 2M + 1$ and $q^2 + q + 1 = 4M^2 + 6M + 3$. Then equality (3) results in

$$p^2 = M\,(4M^2 + 6M + 3), \qquad M > 1. \quad (4)$$

The three divisors of p^2 are 1, p, p^2. It is easily seen that none of these divisors satisfies equality (4). Hence $q \not> 3$, and equality (4) is impossible.

Since no prime q exists which satisfies equation (1), the contradiction derived implies that our assumption is false, and the equation $p^4 + q^3 = z^2$ has no solutions as asserted.

This completes the proof of **(a)**.

(b) Suppose

$$p^4 - q^3 = z^2. \quad (5)$$

We shall assume that there exist distinct odd primes p, q, such that (5) has a solution, and reach a contradiction.

Equation (5) implies

$$q^3 = p^4 - z^2 = (p^2)^2 - z^2 = (p^2 - z)(p^2 + z).$$

Denote

$$p^2 - z = q^C, \qquad p^2 + z = q^D, \qquad C < D, \qquad C + D = 3,$$

where C, D are non-negative integers. Then $q^C + q^D$ results in

$$2p^2 = q^C(q^{D-C} + 1). \quad (6)$$

Since p, q are distinct primes, it follows from (6) that $C > 0$ is impossible. Thus $C = 0$. When $C = 0$ then $D = 3$, and (6) yields $2p^2 = q^3 + 1 = q^3 + 1^3$ or

$$2p^2 = (q + 1)(q^2 - q + 1). \quad (7)$$

All odd primes q satisfy:
(i) $q + 1 > 3$, and $q + 1$ is even.
(ii) $q^2 - q + 1 > q + 1$.

The term $2p^2$ has six divisors, namely: 1, 2, p, $2p$, p^2, $2p^2$.
Firstly, by (i) $q + 1 \neq 1, 2$ since $q + 1 > 3$, and $q + 1 \neq p$, p^2 since $q + 1$ is even.
Secondly, when $q + 1 = 2p$, $2p^2$, then respectively $q^2 - q + 1 = p$, 1. But this is in contradiction of (ii) since $q^2 - q + 1 > q + 1$. Thus, $q + 1 \neq 2p$, $2p^2$, and equality (7) is impossible.

Our assumption is therefore false, and the equation $p^4 - q^3 = z^2$ has no solutions as asserted.

This concludes the proof of **(b)**.

The proof of Theorem 2.1 is complete. □

3. Conclusion

In this paper, we have established for distinct odd primes p, q, that the equations $p^x + q^y = z^2$ and $p^x - q^y = z^2$ have no solutions when $x = 4$ and $y = 3$. In this case $x + y = 7$. Suppose that p, q also include the even prime 2. When $x + y < 7$, both equations have many solutions such as: $2^4 + 3^2 = 5^2$, $3^2 + 7^1 = 4^2$, $3^4 + 19^1 = 10^2$, $5^3 + 19^1 = 12^2$, $5^5 + 11^1 = 56^2$, $5^5 + 239^1 = 58^2$, $7^5 + 617^1 = 132^2$, $3^3 - 2^1 = 5^2$, $5^2 - 3^2 = 4^2$, $5^3 - 2^2 = 11^2$, $7^2 - 13^1 = 6^2$, $7^3 - 19^1 = 18^2$. When $x + y > 7$, we have $3^7 + 313^1 = 50^2$, and with the prime 2: $2^7 + 41^1 = 13^2$, $2^7 - 7^1 = 11^2$, $2^7 - 47^1 = 9^2$, $2^8 - 31^1 = 15^2$, $2^9 + 17^1 = 23^2$, $2^9 + 113^1 = 25^2$, $2^9 - 431^1 = 9^2$.

It seems that both equations have infinitely many solutions in particular when the prime 2 is included.

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