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On the Diophantine Equation $p^x + (p + 5)^y = z^2$ when $p + 5 = 2^{2u}$

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Abstract. There exist infinitely many primes p which satisfy the condition $p + 5 = 2^{2u}$ where u is a positive integer. For all primes p where $p + 5 = 2^{2u}$, and positive integers x, y, z, it is established that the equation $p^x + (p + 5)^y = z^2$ has no solutions.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds. Among them are for example [2, 3, 5, 9].

Articles of various authors have been written on the equation $p^x + (p + A)^y = z^2$ where $A = 4, 6, 8, 10$, and p, $q = p + A$ are primes. For instance [3, 4, 5, 7]. In this article, we consider the equation of the form $p^x + (p + A)^y = z^2$ where p is a prime and A is an odd value. More precisely, we investigate the equation $p^x + (p + A)^y = z^2$ when $A = 5$ with the condition that $p + 5 = 2^{2u}$ where u is an integer. We will show that when $p + 5 = 2^{2u}$ and x, y, z are positive integers, then the equation $p^x + (p + 5)^y = z^2$ has no solutions.

2. Some properties of $p + 5 = 2^{2u}$

All odd primes p are of the form either $4N + 1$ or $4N + 3$. Since $(4N + 1) + 5 = 2(2N + 3) \neq 2^{2u}$, it follows that if $p + 5 = 2^{2u}$ then $p = 4N + 3$.

There exist infinitely many primes $p = 4N + 3$ which satisfy the equality $p + 5 = 2^{2u}$ where u is a positive integer. The first five such primes are $p = 11, 59, 251, 1019, 4091$, and

$11 + 5 = 2^4, \quad 59 + 5 = 2^6, \quad 251 + 5 = 2^8, \quad 1019 + 5 = 2^{10}, \quad 4091 + 5 = 2^{12}.$

The power $2^{2u} = (2^u)^2$ is an even square. As such, it can never end in a last digit equal to 2 or equal to 8. Moreover, it can clearly not end in the digit 0. Therefore, it can end either in the digit 4 or in the digit 6. In which case, p ends in the digit 1 or in the digit 9. Indeed, the primes p presented above end in the digit 1 or in the digit 9.

3. On solutions of $p^x + (p + 5)^y = z^2$ when $p + 5 = 2^{2u}$

In the following theorem, we establish that the equation $p^x + (p + 5)^y = z^2$ has no solutions when $p + 5 = 2^{2u}$ and x, y, z are positive integers.

Theorem 3.1. Suppose that $p \ge 11$ is a prime of the form $4N + 3$, and $p + 5 = 2^{2u}$. Let $u \ge 2$, $m \ge 0$, $n \ge 0$, $z > 0$ be integers.

(a) If $x = 2m$, $y = 2n$, then $p^x + (p + 5)^y = z^2$ has no solutions.
(b) If $x = 2m$, $y = 2n + 1$, then $p^x + (p + 5)^y = z^2$ has no solutions.
(c) If $x = 2m + 1$, $y = 2n$, then $p^x + (p + 5)^y = z^2$ has no solutions.
(d) If $x = 2m + 1$, $y = 2n + 1$, then $p^x + (p + 5)^y = z^2$ has no solutions.

Although quite some similarities exist in the following proofs of these parts, for the convenience of the readers, and for those who are interested in particular parts only, we shall consider these four parts separately, each of which is self-contained.

Proof: **(a)** Suppose that $x = 2m$ $(m \ge 1)$ and $y = 2n$ $(n \ge 1)$.

We shall assume that there exist values p, m, u, n, z such that the equation $p^x + (p + 5)^y = z^2$ has a solution and reach a contradiction.

By our assumption we have

$$p^x + (p + 5)^y = p^{2m} + (p + 5)^{2n} = z^2. \tag{1}$$

From (1) we obtain

$$(p + 5)^{2n} = z^2 - p^{2m} = z^2 - (p^m)^2 = (z - p^m)(z + p^m).$$

Denote

$$z - p^m = (p + 5)^A, \qquad z + p^m = (p + 5)^B, \qquad A < B, \qquad A + B = 2n,$$

where A, B are non-negative integers. Then $(p + 5)^B - (p + 5)^A$ yields

$$2p^m = (p + 5)^A\big((p + 5)^{B - A} - 1\big). \tag{2}$$

In (2), $2 \nmid \big((p + 5)^{B - A} - 1\big)$, hence $2 \mid (p + 5)^A$ and $A > 0$. Since $p + 5 = 2^{2u}$, then $(p + 5)^A = (2^{2u})^A$ where $2uA > 1$. It now follows that the two sides of (2) are contradictory. Thus (2) is impossible.

Our assumption is therefore false, and part **(a)** is complete.

(b) Suppose that $x = 2m$ $(m \ge 1)$ and $y = 2n + 1$ $(n \ge 0)$.

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We shall assume that there exist values p, m, u, n, z such that the equation $p^x + (p + 5)^y = z^2$ has a solution and reach a contradiction.

By our assumption we obtain

$$p^x + (p + 5)^y = p^{2m} + (p + 5)^{2n + 1} = z^2. \tag{3}$$

From (3) we have

$$(p + 5)^{2n + 1} = z^2 - p^{2m} = z^2 - (p^m)^2 = (z - p^m)(z + p^m).$$

Denote

$$z - p^m = (p + 5)^C, \qquad z + p^m = (p + 5)^D, \qquad C < D, \qquad C + D = 2n + 1,$$

where C, D are non-negative integers. Then $(p + 5)^D - (p + 5)^C$ implies

$$2p^m = (p + 5)^C\big((p + 5)^{D - C} - 1\big). \tag{4}$$

In (4), $2 \nmid \big((p + 5)^{D - C} - 1\big)$, thus $2 \mid (p + 5)^C$ and $C > 0$. Since $p + 5 = 2^{2u}$ then $(p + 5)^C = (2^{2u})^C$ where $2uC > 1$. It now follows that the two sides of (4) contradict each other. Hence (4) is impossible.

Our assumption is false, and part **(b)** follows.

(c) Suppose that $x = 2m + 1$ $(m \geq 0)$ and $y = 2n$ $(n \geq 1)$.

We shall assume that there exist values p, m, u, n, z such that the equation $p^x + (p + 5)^y = z^2$ has a solution and reach a contradiction.

By our assumption we have

$$p^x + (p + 5)^y = p^{2m+1} + (p + 5)^{2n} = z^2. \tag{5}$$

From (5) we obtain

$$p^{2m + 1} = z^2 - (p + 5)^{2n} = z^2 - ((p + 5)^n)^2 = (z - (p + 5)^n)(z + (p + 5)^n).$$

Denote

$$z - (p + 5)^n = p^E, \qquad z + (p + 5)^n = p^F, \qquad E < F, \qquad E + F = 2m + 1,$$

where E, F are non-negative integers. Then $p^F - p^E$ yields $2(p + 5)^n = p^E(p^{F - E} - 1)$. Since $p + 5 = 2^{2u}$, we have

$$2^{2un + 1} = p^E(p^{F - E} - 1). \tag{6}$$

The factor p^E divides the right side of (6), but when $E > 0$ $p^E \nmid 2^{2un + 1}$. Therefore $E = 0$ in (6), and accordingly $F = 2m + 1$. Thus (6) yields

$$2^{2un + 1} = p^{2m + 1} - 1. \tag{7}$$

Since $p = 4N + 3$, then for all values $m \geq 0$, the power $p^{2m + 1}$ is of the form $4V + 3$. For all values u, n, (7) yields that

$$2^{2un + 1} = p^{2m + 1} - 1 = (4V + 3) - 1 = 4V + 2 = 2(2V + 1)$$

a contradiction. Therefore (7) is impossible.

Our assumption is false, and the proof of **(c)** is complete.

(d) Suppose that $x = 2m + 1$ $(m \geq 0)$ and $y = 2n + 1$ $(n \geq 0)$.

We shall assume that there exist values p, m, u, n, z such that the equation $p^x + (p + 5)^y = z^2$ has a solution and reach a contradiction.

By our assumption we obtain

$$p^x + (p + 5)^y = p^{2m+1} + (p + 5)^{2n+1} = z^2. \qquad (8)$$

From (8) we have

$$p^{2m+1} = z^2 - (p + 5)^{2n+1} = z^2 - (2^{2u})^{2n+1} = z^2 - (2^{(2n+1)u})^2 = (z - 2^{(2n+1)u})\,(z + 2^{(2n+1)u}).$$

Denote

$$z - 2^{(2n+1)u} = p^G, \quad z + 2^{(2n+1)u} = p^H, \quad G < H, \quad G + H = 2m + 1,$$

where G, H are non-negative integers. Then $p^H - p^G$ yields

$$2 \cdot 2^{(2n+1)u} = p^G(p^{H-G} - 1). \qquad (9)$$

The factor p^G divides the right side of (9). When $G > 0$, then $p^G \nmid 2^{(2n+1)u+1}$. Hence, $G = 0$ in (9), and $H = 2m + 1$. Thus (9) yields

$$2^{(2n+1)u+1} = p^{2m+1} - 1. \qquad (10)$$

The prime p is of the form $p = 4N + 3$. Therefore, for all values $m \geq 0$, the power p^{2m+1} has the form $4Q + 3$. For all values u, n, we then obtain from (10) that

$$2^{(2n+1)u+1} = p^{2m+1} - 1 = (4Q + 3) - 1 = 4Q + 2 = 2(2Q + 1)$$

a contradiction.

Thus (10) is impossible, and our assumption is false.

This concludes the proof of **(d)** and of Theorem 3.1. □

Final Remark. In this article, we have shown for all primes p which satisfy the condition $p + 5 = 2^{2u}$ where u is an integer, that the equation $p^x + (p + 5)^y = z^2$ has no solutions in positive integers x, y, z. The results in this paper were achieved via elementary arguments.

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