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On the Diophantine Equations $2^x + 5^y = z^2$ and $7^x + 11^y = z^2$

Nechemia Burshtein

117 Arlozorov Street, Tel – Aviv 6209814, Israel
Email: anb17@netvision.net.il

Dedicated to Ali Burshtein

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Abstract. In this article, we consider the two equations $2^x + 5^y = z^2$ and $7^x + 11^y = z^2$ when x, y, z are positive integers. The first equation consists of one odd prime, the second equation consists of two odd primes. The equation $2^x + 5^y = z^2$ was already studied by Acu [1]. The purpose of this article is to provide a new approach, shedding new light on the process of finding solutions to equations of the form $p^x + q^y = z^2$. We offer a complete new view relying totally on basic and very elementary arguments.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation
$$p^x + q^y = z^2$$
has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds. Among them are for example [1, 2, 3, 8, 9, 11].

In this article, we consider the two equations
$$2^x + 5^y = z^2,$$
$$7^x + 11^y = z^2.$$

Acu [1] investigated the solutions of the equation $2^x + 5^y = z^2$ when x, y, z are non-negative integers. He obtained the two solutions $(x, y, z) = (3, 0, 3)$ and $(2, 1, 3)$. In this paper, the equation $2^x + 5^y = z^2$ with positive integers x, y, z is considered. Without Catalan's Conjecture as in [1], but with a new elementary technique and basic facts, we establish the unique solution $(x, y, z) = (2, 1, 3)$. Our method totally relies on the last digits of the powers 2^x and 5^y. With positive integers x, y, z, and the same argumentation used before, it is shown that the equation $7^x + 11^y = z^2$ has no solutions.

This is respectively done in Sections 2 and 3. The theorems in each section are self-contained.

2. The solutions of the equation $2^x + 5^y = z^2$

In this section, we consider the equation $2^x + 5^y = z^2$ when x, y, z are positive integers. We shall establish that the equation has a unique solution by using elementary methods only.

First we observe some trivial facts concerning the equation. For all values x, the power 2^x has a last digit which is equal to one of the values 2, 4, 6, 8. For all values y, the power 5^y has a last digit which is equal to 5. The square z^2 is always odd. Therefore z^2 cannot have a last digit which isequal to 3 or equal to 7. These facts and the last digits of the powers 2^x, 5^y are summarized in the following Table 1.

Table 1.

case	last digit of 2^x	last digit of 5^y	last digit of $2^x + 5^y$	solutions of $2^x + 5^y = z^2$
1	2	5	7	no solutions
2	4	5	9	to be determined
3	6	5	1	to be determined
4	8	5	3	no solutions

We now investigate the two undetermined cases 2 and 3. This is done in the respective Theorems 2.1 and 2.2. Both theorems are self-contained.

Theorem 2.1. If $2^x + 5^y$ ends in the digit 9, then $2^x + 5^y = z^2$ has a unique solution.

Proof: If 2^x ends in the digit 4, then $x = 2 + 4m$ where $m \geq 0$ is an integer. When $m = 0$ then $x = 2$, and with $y = 1, z = 3$, we obtain

Solution 1. $$2^2 + 5^1 = 3^2 = z^2.$$

One could easily see for $m = 0$, that this is the only possible solution.

Let $m \geq 1$. We shall assume that for some value m, the equation $2^{2+4m} + 5^y = z^2$ has a solution and reach a contradiction.

By our assumption, we have $2^{2+4m} + 5^y = z^2$ or
$$5^y = z^2 - 2^{2+4m} = z^2 - 2^{2(1+2m)} = (z - 2^{1+2m})(z + 2^{1+2m}).$$
Denote
$$z - 2^{1+2m} = 5^A, \qquad z + 2^{1+2m} = 5^B, \qquad A < B, \qquad A + B = y,$$
where A, B are non-negative integers. Then $5^B - 5^A$ yields
$$2 \cdot 2^{1+2m} = 5^A(5^{B-A} - 1). \tag{1}$$
If $A > 0$, the power 5^A does not divide the left side of (1), and therefore $A = 0$. When $A = 0$, then $B = y$, and (1) results in

$$2^{2 + 2m} = 5^y - 1. \qquad (2)$$

We shall now consider two cases with respect to y.

If y is even, let $y = 2R$ where $R \geq 1$ is an integer. Then (2) implies

$$2^{2 + 2m} = 5^{2R} - 1 = (5^R)^2 - 1^2 = (5^R - 1)(5^R + 1), \qquad m \geq 1. \qquad (3)$$

It is easily seen that the equality $a(a + 2) = 2^G$ which follows from (3) is valid only when $a = 2$ and $G = 3$. Since $m \geq 1$, (3) is impossible, and $y \neq 2R$.

If y is odd, let $y = 2R + 1$. Then from (2) we obtain

$$2^{2 + 2m} = 5^y - 1^y = (5 - 1)(5^{y-1} + 5^{y-2} + \cdot \cdot \cdot + 5^1 + 1). \qquad (4)$$

In (4), the value $(5 - 1) = 2^2 < 2^{2 + 2m}$ since $m \geq 1$, whereas the factor $(5^{y-1} + 5^{y-2} + \cdot \cdot \cdot + 5^1 + 1)$ is an odd integer since it contains $y = 2R + 1$ odd integers. Thus (4) is impossible, and $y \neq 2R + 1$.

For all values $m \geq 1$, we have shown that there does not exist a value y which satisfies (2). This contradiction implies that our assumption that for some value $m \geq 1$, the equation $2^{2 + 4m} + 5^y = z^2$ has a solution is false. As for $m = 0$, $2^{2 + 4m} + 5^y = z^2$ has exactly one solution, namely **Solution 1**.

The proof of Theorem 2.1 is complete. □

In the following theorem we consider case 3 in Table 1.

Theorem 2.2. If $2^x + 5^y$ ends in the digit 1, then $2^x + 5^y = z^2$ has no solutions.

Proof: From Table 1, the power 2^x ends in the digit 6. Then $x = 4m$ where $m \geq 1$ is an integer. We shall assume that for some value m, the equation $2^{4m} + 5^y = z^2$ has a solution and derive a contradiction.

By our assumption, we have $2^{4m} + 5^y = z^2$ or

$$5^y = z^2 - 2^{4m} = z^2 - (2^{2m})^2 = (z - 2^{2m})(z + 2^{2m}).$$

Denote

$$z - 2^{2m} = 5^C, \qquad z + 2^{2m} = 5^D, \qquad C < D, \qquad C + D = y,$$

where C, D are non-negative integers. Then $5^D - 5^C$ results in

$$2 \cdot 2^{2m} = 5^C(5^{D-C} - 1). \qquad (5)$$

If $C > 0$, the power 5^C does not divide the left side of (5), and thus $C = 0$. When $C = 0$, then $D = y$, and (5) yields

$$2^{2m+1} = 5^y - 1. \qquad (6)$$

For all values $m \geq 1$, the last digit of 2^{2m+1} ends either in the digit 8 or in the digit 2. Thus from (6), $2^{2m+1} + 1$ ends either in the digit 9 or in the digit 3, but not in the digit 5 as required. Hence $2^{2m+1} \neq 5^y - 1$. This contradicts our assumption that for some value m, the equation $2^{4m} + 5^y = z^2$ has a solution.

When $2^x + 5^y$ ends in the digit 1, the equation $2^x + 5^y = z^2$ has no solutions.

This concludes the proof of Theorem 2.2. □

Remark 2.1. In [1], using Catalan's Conjecture, Acu found that the equation $2^x + 5^y = z^2$ has the unique solution $(x, y, z) = (2, 1, 3)$ namely **Solution 1**. In this section, via basic facts only with a new elementary technique, **Solution 1** has been established.

3. The solutions of the equation $7^x + 11^y = z^2$

In this section, we consider the equation $7^x + 11^y = z^2$ when x, y, z are positive integers. We shall establish that the equation has no solutions.

Some basic facts on the equation are: The power 7^x has a last digit which is equal to one of the values 1, 3, 7, 9. For all values y, the power 11^y has a last digit equal to 1. The square z^2 is always even, and as such cannot have a last digit equal to 2 or equal to 8. These facts and the last digits of the powers 7^x, 11^y are now summed up in the following Table 2.

Table 2.

case	**last digit of 7^x**	**last digit of 11^y**	**last digit of $7^x + 11^y$**	**solutions of $7^x + 11^y = z^2$**
1	1	1	2	no solutions
2	3	1	4	to be determined
3	7	1	8	no solutions
4	9	1	0	to be determined

In accordance with Table 2, we shall now investigate the two undetermined cases 2 and 4. This is done respectively in the self-contained Theorems 3.1 and 3.2.

Theorem 3.1. If $7^x + 11^y$ ends in the digit 4, then $7^x + 11^y = z^2$ has no solutions.

Proof: If 7^x ends in the digit 3, then $x = 3 + 4m$ where $m \geq 0$ is an integer. We shall assume that for some value m, the equation $7^{3+4m} + 11^y = z^2$ has a solution and reach a contradiction.

Since z^2 is even, therefore $z = 2T$ where T is an integer, and $z^2 = 4T^2$. The primes 7 and 11 are of the form $4N + 3$. For all values m the power 7^{3+4m} is of the form $4V + 3$. In order that $7^{3+4m} + 11^y = 4T^2$, it follows that 11^y clearly has the form $4Q + 1$. This is possible only when y is even. Let $y = 2K$ $(K \geq 1)$ where K is an integer.

We then have $7^{3+4m} + 11^{2K} = z^2$ or

$$7^{3+4m} = z^2 - 11^{2K} = z^2 - (11^K)^2 = (z - 11^K)(z + 11^K). \quad (7)$$

Denote in (7)

$$z - 11^K = 7^E, \qquad z + 11^K = 7^F, \qquad E < F, \qquad E + F = 3 + 4m,$$

where E, F are non-negative integers. Then $7^F - 7^E$ yields

$$2 \cdot 11^K = 7^E(7^{F-E} - 1). \quad (8)$$

In (8) $7 \nmid 2 \cdot 11^K$, and therefore $E \not> 0$. Hence $E = 0$ and $F = 3 + 4m$. Then (8) implies

$$2 \cdot 11^K = 7^{3+4m} - 1. \quad (9)$$

One could easily show for each and every value $m = 0, 1, 2, \ldots$, that $7^{3+4m} - 1$ is a multiple of 3. But in (9), $3 \nmid 2 \cdot 11^K$. Therefore $y \neq 2K$, and $2 \cdot 11^K \neq 7^{3+4m} - 1$.

We have shown that there does not exist a value y which satisfies the equation $7^{3+4m} + 11^y = z^2$. Hence, our assumption that for some value m, the equation $7^{3+4m} + 11^y = z^2$ has a solution is false.

The equation $7^{3+4m} + 11^y = z^2$ has no solutions.

The proof of Theorem 3.1 is complete. □

Theorem 3.2. If $7^x + 11^y$ ends in the digit 0, then $7^x + 11^y = z^2$ has no solutions.

Proof: If 7^x ends in the digit 9, then $x = 2 + 4m$ where $m \geq 0$ is an integer. We shall assume that for some value m, the equation $7^{2+4m} + 11^y = z^2$ has a solution and derive a contradiction.

By our assumption, we have $7^{2+4m} + 11^y = z^2$ or

$$11^y = z^2 - 7^{2(2m+1)} = z^2 - (7^{2m+1})^2 = (z - 7^{2m+1})(z + 7^{2m+1}).$$

Denote

$$z - 7^{2m+1} = 11^G, \qquad z + 7^{2m+1} = 11^H, \qquad G < H, \qquad G + H = y,$$

where G, H are non-negative integers. Then $11^H - 11^G$ results in

$$2 \cdot 7^{2m+1} = 11^G(11^{H-G} - 1). \quad (10)$$

If $G > 0$, then $11 \nmid 2 \cdot 7^{2m+1}$ in (10). Hence $G = 0$ and accordingly $H = y$. Thus (10) yields

$$2 \cdot 7^{2m+1} = 11^y - 1. \quad (11)$$

Since for all values y, 11^y has a last digit equal to 1, hence $11^y - 1$ has a last digit equal to 0. It therefore follows in (11) that $11^y - 1$ is a multiple of 5. But the left side of (11) is not a multiple of 5. Hence, $2 \cdot 7^{2m+1} \neq 11^y - 1$.

Our assumption that for some value m, the equation $7^{2+4m} + 11^y = z^2$ has a solution is therefore false.

The equation $7^{2+4m} + 11^y = z^2$ has no solutions.

This concludes the proof of Theorem 3.2. □

Remark 3.1. It has been shown that cases 2 and 4 of Table 2 do not yield solutions to the equation $7^x + 11^y = z^2$. Together with cases 1 and 3 of Table 2, it is now established that the equation $7^x + 11^y = z^2$ with positive integers x, y, z has no solutions.

Final remark. In our arguments, we have employed only basic and elementary methods to establish that $2^x + 5^y = z^2$ has a unique solution, and that $7^x + 11^y = z^2$ has no solutions when x, y, z are positive integers. The results were achieved by a new elementary technique which uses the last digits of the powers involved, namely 2^x, 5^y, 7^x, 11^y together with other basic simple properties.

We may suggest that this new elementary technique be used in other equations in the process of finding solutions.

Referring to Euclid's elementary yet ingenious proof of the infinitude of primes, an anonymous once said: "the beauty of a proof lies in its simplicity".

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