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The Classification of Some Order Four Digraphs As Nonnegative $P_{0,1}$-Matrix Completion

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Abstract. A digraph of order n is said to have nonnegative $P_{0,1}$-completion if every partial nonnegative $P_{0,1}$-matrix that specifies the digraph can be completed to a nonnegative $P_{0,1}$-matrix. The aim of this paper is to classify some digraphs of order four that include all loops as either having nonnegative $P_{0,1}$-completion or not having nonnegative $P_{0,1}$-completion.

1. Introduction
In this section, we recall and introduce some of the definitions and fundamental concepts that play a key role in this paper.

Definition 1.1. A real $n \times n$ matrix A is a P (resp. P_0)-**matrix** if all principal minor of the matrix A are positive (resp. nonnegative). A P_0-matrix A with all its diagonal entries being positive is called a $P_{0,1}$-**matrix**.

Definition 1.2. A **nonnegative** P (resp. $P_0, P_{0,1}$)-**matrix** is a P (resp. $P_0, P_{0,1}$)-matrix in which all entries are nonnegative.

Definition 1.3. A **partial matrix** is a matrix in which some entries are specified and others are not specified.

Let π be P, P_0, $P_{0,1}$, nonnegative P, nonnegative P_0 or nonnegative $P_{0,1}$ then the following definitions.

Definition 1.4. A **partial π -matrix** is a partial matrix in which all specified entries satisfies the conditions of a π -matrix.

Definition 1.5. A **completion** of a partial matrix is a specific choice of values for the unspecified entries resulting in a required matrix.

During the completion of partial matrices, digraphs are used and in some papers they use patterns which means same thing in matrix completions. Therefore, we have the following useful definitions for patterns and digraphs.

Definition 1.6. A **pattern** Q for $n \times n$ matrices is a list of positions of the specified entries of a matrix. In view of a digraphs, a **digraph** of a pattern for $n \times n$ matrix is the digraph $D = (V_D, A_D)$ with set V_D of vertices equal to $\{1,...,n\}$ and set A_D of arcs equal to the ordered (i, j) such that $i \neq j$ and position (i, j) is in the pattern or in other words, if an arc exist between vertices i and j then we say position (i, j) of the partial matrix is specified.

Definition 1.7. The **order** of a digraph D denoted n is the number of vertices of D. A digraph is **complete digraph** or **clique** if it includes all possible arcs between its vertices, and if patterns then all positions (i, j) are listed. A digraph is called **symmetric** if an arc (i, j) exist if and only if (j,i) is in the graph and is called **asymmetric** if an arc (j,i) is in the graph then (j,i) does not exist in the graph. A **path** in a digraph is a sequence of vertices $v_1,..., v_k$ in V_D such that for $i = 1,..., k-1$, the arc $(v_i, v_{i+1}) \in A_D$ and all vertices are unique except maybe $v_1 = v_k$ and if that is the case then it is called k **-cyclic** digraph [9].

Definition 1.8. A partial matrix A is said to **specify a pattern-digraph** if its specified entries lie exactly in those positions listed in the pattern or for a digraph position a_{ij} of the partial matrix A is specified if and only if there is an arc between vertex v_i and v_j of D.

 Example 1.9. Consider the digraph in Figure 1.
The digraph in Figure 1, is of order 3 with 4 arcs (non-loops), and it is 3-cycle with the path $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_1$.

The Classification of Some Order Four Digraphs as Nonnegative $P_{0,1}$-Matrix Completion
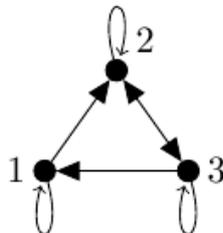
Figure 1: Digraph of order 3

The partial matrix that specifies digraph in Figure 1 is given below.

$$\begin{bmatrix} d_1 & a_{12} & x_{13} \\ x_{21} & d_2 & a_{23} \\ a_{31} & a_{32} & d_3 \end{bmatrix}$$

In the partial matrix d_i denoted the specified diagonal entries, in the digraph is the loops. a_{ij} is the specified entry and in the digraph it represents the arcs (non-loops) and finally, x_{ij} is the unspecified non-diagonal entries and in the digraphs it represents those vertices that do not have arcs between them.

Translating above to a pattern we have, $Q = \{(1,1),(1,2),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

Definition 1.10. A pattern (or its (di)graph) has π-**completion** if every partial π-matrix which specifies the pattern (or its (di)graph) can be completed to a π-matrix.
Note that if a pattern contains all the diagonal entries then the digraph specified contains loops in all vertices.

All non-isomorphic digraphs of order at most four are given in [9], where p denotes number vertices, q denotes the number of arcs and for non-isomorphic digraphs with equal number of vertices and arcs, it has a position number denoted n. Whenever a digraph has only one non-isomorphic digraph then there is no need of using n the digraph number.

In Section 2, we discuss closely related work leading to this problem and in Section 3, we give preliminary results in this paper. The main results on classification of digraphs of order 4 regarding nonnegative $P_{0,1}$-completion are given in Section 4.

2. Related work
Over the past years, several matrix completion problems have been considered. The authors in [1] showed that every symmetric partial P-matrix has P-completion. Another paper on same class [2] proved that any pattern of 3×3 matrices has P-completion. Furthermore, they classified 207 out of 218 digraphs as either having P-completion or not having P-completion. Hogben [3] proved that every digraph of order

3 has nonnegative P-completion and in [4] all digraphs of order 4 were classified regarding nonnegative P-completion.

Another related class is of P_0-matrices and its subclass nonnegative P_0-matrices. It was noted in Example 9.1 of [3] and Example 3.1 of [5] that not all digraphs of order 3 have P_0-completion. Moreover, authors in [5] successfully classified all digraphs of order 4 as either having P_0-completion or not having P_0-completion. A year later, it was shown in [6] that all digraphs of order 3 that includes all loops have nonnegative P_0-completion. In addition, they provided complete classifications of digraphs of order 4 as nonnegative P_0-completion.

The third and closely related class is of $P_{0,1}$-matrices. In 2001, Hogben [3] proved that all digraphs of order 3 have both $P_{0,1}$-completion and nonnegative $P_{0,1}$-completion. Wangsness [7] in her PhD thesis extended the work of Hogben [3], she classified 192 out of 218 digraphs of order 4 regarding $P_{0,1}$-completion. In the present paper, we will classify some digraphs of order 4 as having nonnegative $P_{0,1}$-completion or not having $P_{0,1}$-completion.

3. Preliminary results
In this section, we will present some basic results that will be useful in the next section when classifying digraphs of order four. We first present classifications of digraphs order 2 and 3 in Theorem 3.1 and Theorem 3.2 respectively.

Theorem 3.1. [3, Lemma 4.8] All patterns for 2×2 matrices have nonnegative $P_{0,1}$-completion.

Theorem 3.2: [3, Lemma 8.1] A pattern for 3×3 matrices with all specified diagonal entries has nonnegative $P_{0,1}$-completion.

According to [3, Example 4.3], it was clear that some partial nonnegative $P_{0,1}$-matrices with some unspecified diagonal entries cannot be completed to a nonnegative $P_{0,1}$-matrices. This leads us to some necessary conditions of nonnegative $P_{0,1}$-matrices of having all diagonal entries specified or rather we use digraphs that include all loops.

Lemma 3.3. [3, Theorem 8.4] Any symmetric n-cycle that has loops at every vertex has nonnegative $P_{0,1}$-completion.

Lemma 3.4. [8, Theorem 2.13] Any asymmetric pattern that has non-negative P-completion also has nonnegative $P_{0,1}$-completion.

The Classification of Some Order Four Digraphs as Nonnegative $P_{0,1}$-Matrix Completion

Lemma 3.5. [8, Theorem 2.12] Any pattern that has nonnegative P_0-completion also has nonnegative $P_{0,1}$-completion.

Lemma 3.6. [8, Corollary 2.9] Any pattern that has nonnegative $P_{0,1}$-completion also has nonnegative P-completion.

We will use Lemmas 3.3, 3.4 and 3.5 to list some digraphs of order 4 having nonnegative $P_{0,1}$-completion and Lemma 3.6 will be used to list some digraphs that do not have nonnegative $P_{0,1}$-completion.

4. Main results

We are now able to classify most of digraphs of order 4 as having nonnegative $P_{0,1}$-completion or not having nonnegative $P_{0,1}$-completion.

Theorem 4.1: The following digraphs of order 4 have nonnegative $P_{0,1}$-completion

$$q = 0;$$
$$q = 1;$$
$$q = 2, \quad n = 1\text{-}5;$$
$$q = 3, \quad n = 1\text{-}13;$$
$$q = 4 \quad n = 1\text{-}27;$$
$$q = 5 \quad n = 1\text{-}33,35\text{-}38;$$
$$q = 6 \quad n = 1\text{-}21,23\text{-}32,35,36,38\text{-}41,43,44,46\text{-}48;$$
$$q = 7 \quad n = 1-7,13\text{-}17,19,21\text{-}24,26,28,29,31,34,36,37;$$
$$q = 8 \quad n = 1\text{-}2,6,10,12,14,15,18,21,27;$$
$$q = 9 \quad n = 8,11;$$
$$q = 12$$

Proof:

$q = 4, n = 16 \qquad q = 5, n = 7 \qquad q = 6, n = 4 \qquad q = 6, n = 7 \qquad q = 7, n = 2 \qquad q = 8, n = 2$
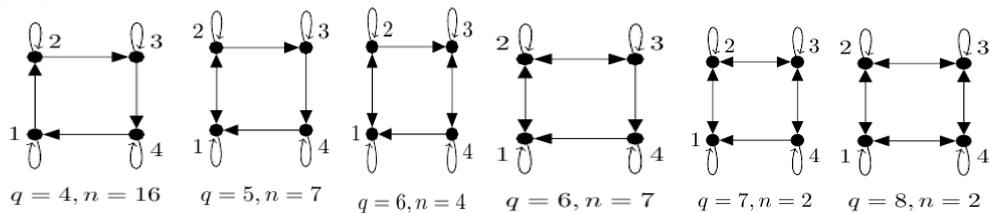
Digraph $q=8$ $n=2$ is a symmetric 4-cycle and in view of Lemma 3.3, it has nonnegative $P_{0,1}$-completion.

A partial nonnegative $P_{0,1}$-matrix specifying any of the digraphs $q=4$ $n=16$; $q=5$ $n=7$; $q=6$ $n=4,7$ and $q=7$ $n=2$ may be extended to a partial nonnegative $P_{0,1}$-matrix specifying $q=8$ $n=2$ by setting all unspecified entries to zeros. Therefore, all these digraphs have nonnegative $P_{0,1}$-completion.
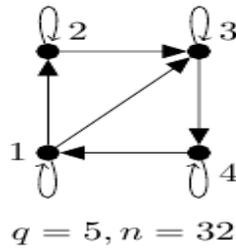
$$q = 5, n = 32$$

Digraph $q=5$ $n=32$ is an asymmetric digraph and it has nonnegative P-completion by Theorem 2.6 of [5]. Thus it also has nonnegative $P_{0,1}$-completion by Lemma 3.4.

According to [6, Lemmas 2.2, 2.3, 2.4 and 2.5], all the remaining digraphs listed above have nonnegative P_0-completion. By Lemma 3.5, they also have nonnegative $P_{0,1}$-completion.

Observe also that digraph $q=4$ $n=16$ is asymmetric and has nonnegative P-completion then one can still use Lemma 3.4 to proof that it has nonnegative $P_{0,1}$-completion.

Theorem 4.2. The following digraphs of order 4 do not have nonnegative $P_{0,1}$-completion

$q=6$ $n=45$;

$q=7$ $n=30,32,33,35,38$;

$q=8$ $n=16,17,19,20,22$-27;

$q=9$ $n=3$-$7,9,10,12,13$;

$q=10$ $n=1-5$;

$q=11$.

Proof: According to [4, Theorem 2.6], each of the digraphs listed do not have nonnegative P-completion. Hence, by Lemma 3.6 these digraphs do not have nonnegative $P_{0,1}$-completion.

Remark 4.3. The main focus of this paper was to classify digraphs of order four as nonnegative $P_{0,1}$-completion. However, the following 23 digraphs were not classified

The Classification of Some Order Four Digraphs as Nonnegative $P_{0,1}$-Matrix Completion

and one may study their completeability and classify them as having nonnegative $P_{0,1}$-completion or not having nonnegative $P_{0,1}$-completion.

$$q = 6 \qquad n = 22,33,34,37,42;$$
$$q = 7 \qquad n = 8,10,12,13,18,20,25,27;$$
$$q = 8 \qquad n = 3\text{-}5,7\text{-}9,11,13;$$
$$q = 9 \qquad n = 1-2;$$

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