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Strong Domination Parameters in Operations on Fuzzy Graphs

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Abstract. In this paper, the concept of domination is introduced on various operations on fuzzy graphs such as union, join, composition and cartesian product using strong arcs. The different types of strong domination parameters of union and join of two fuzzy graphs is obtained.It is obtained the strong domination set of cartesian product of two fuzzy graphs.

1. Introduction
Fuzzy graphs were introduced by Rosenfeld, who has described the fuzzy analogue of several graph theoretic concepts like paths, cycles, trees and connectedness [19]. Bhutani and Rosenfeld have introduced the concept of strong arcs [2].

The work on fuzzy graphs was also done by Mordeson, Pradip, Talebi,Yeh, Michael, sunil and Pathinathan [13, 18, 29, 30, 31, 32, 33]. It was during 1850s, a study of dominating sets in graphs started purely as a problem in the game of chess. Chess enthusiasts in Europe considered the problem of determining the minimum number of queens that can be placed on a chess board so that all the squares are either attacked by a queen or occupied by a queen. The concept of domination in graphs was introduced by Ore and Berge in 1962, the domination number and independent domination number are introduced by Cockayne and Hedetniemi [6]. Connected domination in graphs was discussed by Sampathkumar and Walikar [20]. Somasundaram and Somasundaram discussed domination in fuzzy graphs. They defined domination using effective edges in fuzzy graph [22, 23]. Nagoorgani and Chandrasekharan defined domination in fuzzy graphs using strong arcs [15]. Manjusha and Sunitha discussed some concepts in domination and total domination in fuzzy graphs using strong arcs [7, 8]. In this paper, we study various operations on fuzzy graphs such as union,join, composition and cartesian product and study their domination parameters using strong arcs.

2. Preliminaries
We summarize briefly some basic definitions in fuzzy graphs which are presented in [1, 2,

4, 14, 15, 19, 22, 25].

A **fuzzy graph** is denoted by $G: (V, \sigma, \mu)$ where V is a node set, σ is a fuzzy subset of V and μ is a fuzzy relation on σ. i.e., $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$. We call σ the fuzzy node set of G and μ the fuzzy arc set of G, respectively. We consider fuzzy graph G with no loops and assume that V is finite and nonempty, μ is reflexive (i.e., $\mu(x, x) = \sigma(x)$, for all x) and symmetric (i.e., $\mu(x, y) = \mu(y, x)$, for all (x, y)). In all the examples σ is chosen suitably. Also, we denote the underlying crisp graph by $G^*: (\sigma^*, \mu^*)$ where $\sigma^* = \{u \in V : \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in V \times V : \mu(u, v) > 0\}$. Throughout we assume that $\sigma^* = V$. The fuzzy graph $H: (\tau, v)$ is said to be a **partial fuzzy subgraph** of $G: (\sigma, \mu)$ if $v \subseteq \mu$ and $\tau \subseteq \sigma$. In particular we call $H: (\tau, v)$, a fuzzy subgraph of $G: (\sigma, \mu)$ if $\tau(u) = \sigma(u)$ for all $u \in \tau^*$ and $v(u, v) = \mu(u, v)$ for all $(u, v) \in v^*$. A fuzzy subgraph $H: (\tau, v)$ spans the fuzzy graph $G: (V, \sigma, \mu)$ if $\tau = \sigma$. The fuzzy graph $H: (P, \tau, v)$ is called an induced fuzzy subgraph of $G: (V, \sigma, \mu)$ induced by P if $P \subseteq V$ and $\tau(u) = \sigma(u)$ for all $u \in P$ and $v(u, v) = \mu(u, v)$ for all $u, v \in P$. We shall use the notation $< P >$ to denote the fuzzy subgraph induced by P. $G: (V, \sigma, \mu)$ is called trivial if $|\sigma^*| = 1$.

In a fuzzy graph $G: (V, \sigma, \mu)$, a **path** P of length n is a sequence of distinct nodes $u_0, u_1, \ldots, u_n$ such that $\mu(u_{i-1}, u_i) > 0, i = 1, 2, \ldots, n$ and the degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n$ and $n \geq 3$ then P is called a cycle and P is called a fuzzy cycle, if it contains more than one weakest arc. The strength of a cycle is the strength of the weakest arc in it. The strength of connectedness between two nodes x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by $CONN_G(x, y)$.

A fuzzy graph $G: (\sigma, \mu)$ is **connected** if for every x, y in $\sigma^*, CONN_G(x, y) > 0$. An arc (u, v) of a fuzzy graph is called an **effective arc** (M-strong arc) if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$. Then u and v are called effective neighbors. The set of all effective neighbors of u is called effective neighborhood of u and is denoted by $EN(u)$.

A fuzzy graph G is said to be **complete** if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$, for all $u, v \in \sigma^*$ and is denoted by K_σ.

The **order** p and **size** q of a fuzzy graph $G: (\sigma, \mu)$ are defined to be $p = \sum_{x \in V} \sigma(x)$ and $q = \sum_{(x,y) \in V \times V} \mu(x, y)$.

Let $G: (V, \sigma, \mu)$ be a fuzzy graph and $S \subseteq V$. Then the **scalar cardinality** of S is defined to be $\sum_{v \in S} \sigma(v)$ and it is denoted by $|S|$. Let p denotes the scalar cardinality of V, also called the order of G.

An arc of a fuzzy graph is called **strong** if its weight is at least as great as the strength of connectedness of its end nodes when it is deleted. Depending on $CONN_G(x, y)$ of an arc (x, y) in a fuzzy graph G, Sunil Mathew and Sunitha [25] defined three different types of arcs. Note that $CONN_{G-(x,y)}(x, y)$ is the the strength of connectedness between x and y in the fuzzy graph obtained from G by deleting the arc (x, y). An arc (x, y) in G is $\boldsymbol{\alpha - \text{strong}}$ if $\mu(x, y) > CONN_{G-(x,y)}(x, y)$. An arc (x, y) in G is $\boldsymbol{\beta - \text{strong}}$ if $\mu(x, y) = CONN_{G-(x,y)}(x, y)$. An arc (x, y) in G is $\boldsymbol{\delta - \text{arc}}$ if $\mu(x, y) < CONN_{G-(x,y)}(x, y)$.

Thus an arc (x, y) is a strong arc if it is either $\alpha - $ strong or $\beta - $ strong. A path P is called strong path if P contains only strong arcs.

A fuzzy graph G is said to be **bipartite** [22] if the node set V can be partitioned

into two non empty sets V_1 and V_2 such that $\mu(v_1, v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$. Further if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all $u \in V_1$ and $v \in V_2$ then G is called a **complete bipartite graph** and is denoted by K_{σ_1, σ_2}, where σ_1 and σ_2 are respectively the restrictions of σ to V_1 and V_2.

A connected fuzzy graph $G = (V, \sigma, \mu)$ is called a **fuzzy tree** if it has a fuzzy spanning subgraph $F: (\sigma, \nu)$, which is a tree [spanning tree], where for all arcs (x, y) not in F there exists a path from x to y in F whose strength is more than $\mu(x, y)$ [19]. Note that here F is a tree which contains all nodes of G and hence is a spanning tree of G.

A **maximum spanning tree** of a connected fuzzy graph $G: (V, \sigma, \mu)$ is a fuzzy spanning subgraph $T: (\sigma, \nu)$, such that T is a tree, and for which $\sum_{u \neq v} \nu(u, v)$ is maximum. A node which is not an endnode of T is called an internal node of T [14].

A node u is said to be **isolated** if $\mu(u, v) = 0$ for all $v \neq u$.

3. Strong domination parametrs in operations on fuzzy graphs

The concept of domination in operations on fuzzy graphs was introduced by Somasundaram [24]. The author used effective arcs in this concept. This study leads to group all the fuzzy graphs without effective arcs under one category. Also note that each effective arc is strong, but a strong arc need not be effective [2]. Hence a generalized study of domination parameters in operations on fuzzy graphs using strong arcs is more relevant. Let us recall some basic definitions of domination parameters and operations on fuzzy graphs.

Definition 1. [15] A node v in a fuzzy graph G is said to strongly dominate itself and each of its strong neighbors, i.e., v strongly dominates the nodes in $N_s[v]$. A set D of nodes of G is a strong dominating set of G if every node of $V(G) - D$ is a strong neighbor of some node in D.

Definition 2. [9] The weight of a strong dominating set D is defined as $W(D) = \sum_{u \in D} \mu(u, v)$, where $\mu(u, v)$ is the minimum of the membership values(weights) of strong arcs incident on u. The strong domination number of a fuzzy graph G is defined as the minimum weight of strong dominating sets of G and it is denoted by $\gamma_s(G)$ or simply γ_s. A minimum strong dominating set in a fuzzy graph G is a strong dominating set of minimum weight.

Definition 3. [8] A set D of nodes in a fuzzy graph $G: (V, \sigma, \mu)$ is a strong total(open) dominating set of G if every node of G is a strong neighbor of at least one node of D.

Definition 4. [8] The weight of a strong total dominating set D is defined as $W(D) = \sum_{u \in D} \mu(u, v)$, where $\mu(u, v)$ is the minimum of the weight of the strong arcs incident on u. The strong total domination number of a fuzzy graph G is defined as the minimum weight of strong total dominating sets of G and it is denoted by $\gamma_{st}(G)$ or simply γ_{st}. A minimum strong total dominating set in a fuzzy graph G is a strong total dominating set of minimum weight.

Definition 5. [10] A strong dominating set D of a fuzzy graph $G: (V, \sigma, \mu)$ is a strong connected dominating set of G if the induced fuzzy subgraph $< D >$ is connected.

Definition 6. [10] The weight of a strong connected dominating set D is defined as $W(D) = \sum_{u \in D} \mu(u, v)$, where $\mu(u, v)$ is the minimum of the membership values(weights) of strong arcs incident on u. The strong connected domination number of a fuzzy graph G is defined as the minimum weight of strong connected dominating sets of G and it is denoted by $\gamma_{sc}(G)$ or simply γ_{sc}. A minimum strong connected dominating set in a fuzzy graph G is a strong connected dominating set of minimum weight.

Definition 7. [15] Two nodes in a fuzzy graph $G: (V, \sigma, \mu)$ are said to be **strongly independent** if there is no strong arc between them. A set of nodes in G is **strong independent** if any two nodes in the set are strongly independent.

Definition 8. [15] A strong dominating set D of a fuzzy graph $G: (V, \sigma, \mu)$ is called a **strong independent dominating set** if D is strongly independent.

Definition 9. The **weight of a strong independent dominating set** D is defined as $W(D) = \sum_{u \in D} \mu(u, v)$, where $\mu(u, v)$ is the minimum of the membership values(weights) of the strong arcs incident on u. The strong independent domination number of a fuzzy graph G is defined as the minimum weight of strong independent dominating sets of G and it is denoted by $S_i(G)$ or simply S_i. A minimum strong independent dominating set in a fuzzy graph G is a strong independent dominating set of minimum weight.

Definition 10. [17] Let $G: (V, \sigma, \mu)$ be a fuzzy graph. A set M of strong arcs in G such that no two arcs in M have a common node is called a **strong independent set** of arcs or a **strong matching** in G.

Definition 11. [12] Let M be a strong matching in a fuzzy graph $G: (V, \sigma, \mu)$. If $e = (u, v) \in M$, then we say that M strongly matches u to v. The **weight** of a **strong matching** is defined as $W(M) = \sum_{(u,v) \in M} \mu(u, v)$. The **strong arc independence number** or **strong matching number** of a fuzzy graph G is defined as the maximum weight of strong matchings of G and it is denoted by $\beta_{S_1}(G)$ or simply β_{S_1}. A maximum strong matching in a fuzzy graph G is a strong matching of maximum weight.

Definition 12. [12] Let $G: (V, \sigma, \mu)$ be a fuzzy graph and let M be a strong matching in G. Then M is called **perfect strong matching** if M strongly matches every node of G to some node of G.

Definition 13. [12] Let $G: (V, \sigma, \mu)$ be a fuzzy graph. A set $D \subseteq V$ of nodes is said to be a **strong paired dominating set** if D is a strong dominating set and the induced fuzzy subgraph $< D >$ has a perfect strong matching. The **weight** of a **strong paired dominating set** is defined as $W(D) = \sum_{u \in D} \mu(u, v)$, where $\mu(u, v)$ is the minimum of the weights of strong arcs incident on u. The **strong paired domination number** of a fuzzy graph G is defined as the minimum weight of strong paired dominating sets of G and it is denoted by $\gamma_{spr}(G)$ or simply γ_{spr}. A minimum strong paired dominating set in a fuzzy graph G is a strong paired dominating set of minimum weight.

Definition 14. [13, 14] Union of two fuzzy graphs: Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two fuzzy graphs with $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ with $V_1 \cap V_2 = \phi$ and let $G^* = G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$ be the union of G_1^* and G_2^*. Then the union of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G:(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ defined by

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u) & if \ u \in \ V_1 \backslash \ V_2 \\ \\ \sigma_2(u) & if \ u \in \ V_2 \backslash \ V_1 \end{cases}$$

and

$$(\mu_1 \cup \mu_2)(u,v) = \begin{cases} \mu_1(u,v) & if \ (u,v) \in \ E_1 \backslash \ E_2 \\ \\ \mu_2(u,v) & if \ (u,v) \in \ E_2 \backslash \ E_1 \end{cases}$$

We now proceed to study various domination parameters in union of two fuzzy graphs.

Theorem 15. If G_1 and G_2 are two fuzzy fuzzy graphs on V_1 and V_2 respectively with $V_1 \cap V_2 = \phi$.Then
1. $\gamma_s(G_1 \cup G_2) = \gamma_s(G_1) + \gamma_s(G_2)$
2. If G_1 and G_2 are fuzzy graphs having no isolated nodes, then
$$\gamma_{st}(G_1 \cup G_2) = \gamma_{st}(G_1) + \gamma_{st}(G_2)$$
3. $S_i(G_1 \cup G_2) = S_i(G_1) + S_i(G_2)$
4. If G_1 and G_2 are fuzzy graphs having no isolated nodes, then
$$\gamma_{spr}(G_1 \cup G_2) = \gamma_{spr}(G_1) + \gamma_{spr}(G_2)$$
Proof: Any type strong dominating set D of $G_1 \cup G_2$ is of the form $D = D_1 \cup D_2$, where D_1 is any type strong dominating set of G_1 and D_2 is any type strong dominating set of G_2 It follows that results 1, 2, 3 and 4 hold.

Remark 16. Since $G_1 \cup G_2$ is disconnected $\gamma_{sc}(G_1 \cup G_2)$ is not defined.

Definition 17. [13, 14] Join of two fuzzy graphs:Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two fuzzy graphs with $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ with $V_1 \cap V_2 = \phi$ Consider the join $G^* = G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$ of graphs where E' is the set of all arcs joining the nodes of V_1 and V_2 where we assume that $V_1 \cap V_2 = \phi$. Then the join of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G = G_1 + G_2:(\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ defined by
$$(\sigma_1 + \sigma_2)(u) = (\sigma_1 \cup \sigma_2)(u), \ u \in V_1 \cup V_2$$
and
$$(\mu_1 + \mu_2)(u,v) = \begin{cases} (\mu_1 \cup \mu_2)(u,v) & if \ (u,v) \in \ E_1 \cup \ E_2 \ and \\ \\ \sigma_1(u) \wedge \sigma_2(v) & if \ (u,v) \in \ E' \end{cases}$$
We now proceed to study various domination parameters in join of two fuzzy graphs.

Theorem 18. If G_1 and G_2 are two fuzzy fuzzy graphs on V_1 and V_2 respectively with $V_1 \cap V_2 = \phi$.Then $\gamma_s(G_1 + G_2) = \wedge \ \{W(D), W(\{u,v\}): u \in V_1, v \in V_2\}$, where D is any minimal strong dominating set of G_1 or G_2 which is also a minimal strong dominating set of $G_1 + G_2$.
Proof: From the definition of $G_1 + G_2$ it is clear that any arc of the form (u,v), where $u \in V_1$ and $v \in V_2$ is a strong arc since it is an effective arc. Hence any node of V_1

strongly dominates all the nodes of V_2 and vice-versa. Now, let D be any minimal strong dominating set of $G_1 + G_2$. Then D has the following 3 possibilities.

1. $D = D_1$ where D_1 is a minimal strong dominating set of G_1
2. $D = D_2$ where D_2 is a minimal strong dominating set of G_2
3. $D = \{u, v\}$ where $u \in V_1, v \in V_2$, $\{u\}$ is not a strong dominating set of G_1 and $\{v\}$ is not a strong dominating set of G_2. Hence

$$\gamma_s(G_1 + G_2) = \wedge \quad \{W(D), W(\{u, v\}): u \in V_1, v \in V_2\}.$$

Theorem 19. If G_1 and G_2 are two fuzzy fuzzy graphs on V_1 and V_2 respectively with $V_1 \cap V_2 = \phi$.Then

$$S_i(G_1 + G_2) = \wedge \quad \{W(D)\},$$

where D is any minimal strong independent dominating set of G_1 or G_2 which is also a minimal strong independent dominating set of $G_1 + G_2$.

Proof: Since any node of V_1 strongly dominates every node of V_2 in $G_1 + G_2$, any strong independent set in $G_1 + G_2$ is either a subset of V_1 or a subset of V_2. Hence any minimal strong independent dominating set D of $G_1 + G_2$ is one of the following forms.

1. $D = D_1$, where D_1 is a minimal strong independent dominating set of G_1
2. $D = D_2$, where D_2 is a minimal strong independent dominating set of G_2

Hence $S_i(G_1 + G_2) = \wedge \{W(D)\}$, where D is any minimal strong independent dominating set of G_1 or G_2 which is also a minimal strong independent dominating set of $G_1 + G_2$.

Theorem 20. If G_1 and G_2 are two fuzzy fuzzy graphs on V_1 and V_2 respectively with $V_1 \cap V_2 = \phi$.Then

1. If G_1 and G_2 has no isolated nodes, then

$$\gamma_{st}(G_1 + G_2) = \wedge \quad \{W(D), W(\{u, v\}): u \in V_1, v \in V_2\}$$

where D is any minimal strong total dominating set of G_1 or G_2 which is also a minimal strong total dominating set of $G_1 + G_2$.

2. If G_1 has no isolated nodes and G_2 has an isolated node, then

$$\gamma_{st}(G_1 + G_2) = \wedge \quad \{W(D), W(\{u, v\}): u \in V_1, v \in V_2\}$$

where D is any minimal strong total dominating set of G_1 which is also a minimal strong total dominating set of $G_1 + G_2$.

3.If G_1 has an isolated node and G_2 has no isolated nodes, then

$$\gamma_{st}(G_1 + G_2) = \wedge \quad \{W(D), W(\{u, v\}): u \in V_1, v \in V_2\}$$

where D is any minimal strong total dominating set of G_2 which is also a minimal strong total dominating set of $G_1 + G_2$.

4. If both G_1 and G_2 have isolated nodes, then

$$\gamma_{st}(G_1 + G_2) = \wedge \quad \{W(\{u, v\}): u \in V_1, v \in V_2\}$$

Proof: Note that for any two fuzzy graphs G_1 and G_2, its join $G_1 + G_2$ is a fuzzy graph having no isolated nodes and hence $\gamma_{st}(G_1 + G_2)$ always exists. We have the following cases.

CASE 1: Both G_1 and G_2 have no isolated nodes.

In this case $\gamma_{st}(G_1)$ and $\gamma_{st}(G_2)$ exist. Any minimal strong total dominating set D of $G_1 + G_2$ is one of the following forms.

1. $D = D_1$, where D_1 is a minimal strong total dominating set of G_1.
2. $D = D_2$, where D_2 is a minimal strong total dominating set of G_2.

3. $D = \{u, v\}$, where $u \in V_1$ and $v \in V_2$.

Hence $\gamma_{st}(G_1 + G_2) = \wedge \quad \{W(D), W(\{u, v\}): u \in V_1, v \in V_2\}$

where D is any minimal strong total dominating set of G_1 or G_2 which is also a minimal strong total dominating set of $G_1 + G_2$.

CASE II: G_1 has no isolated nodes and G_2 has an isolated node.

In this case $\gamma_{st}(G_2)$ does not exist. Hence any strong total dominating set of $G_1 + G_2$ has nonempty intersection with V_1. Now, let D be any minimal strong total dominating set of $G_1 + G_2$. Then D is one of the following forms:

1. $D = D_1$, where D_1 is a minimal strong total dominating set of G_1.

2. $D = \{u, v\}$, where $u \in V_1$ and $v \in V_2$.

Hence $\gamma_{st}(G_1 + G_2) = \wedge \quad \{W(D), W(\{u, v\}): u \in V_1, v \in V_2\}$

where D is any minimal strong total dominating set of G_1 which is also a minimal strong total dominating set of $G_1 + G_2$.

CASE III: G_1 has an isolated node and G_2 has no isolated nodes.

Proof is similar to that of CASE II.

CASE IV: Both G_1 and G_2 have isolated nodes.

In this case both $\gamma_{st}(G_1)$ and $\gamma_{st}(G_2)$ do not exist. Hence any strong total dominating set of $G_1 + G_2$ has nonempty intersection with both V_1 and V_2, so that any minimal strong total dominating set of $G_1 + G_2$ is of the form $\{u, v\}$, where $u \in V_1, v \in V_2$. Hence

$$\gamma_{st}(G_1 + G_2) = \wedge \quad \{W(\{u, v\}): u \in V_1, v \in V_2\}$$

Theorem 21. If G_1 and G_2 are two fuzzy fuzzy graphs on V_1 and V_2 respectively with $V_1 \cap V_2 = \phi$. Then

1. If G_1 and G_2 has no isolated nodes, then

$$\gamma_{spr}(G_1 + G_2) = \wedge \quad \{W(D), W(\{u, v\}): u \in V_1, v \in V_2\}$$

where D is any minimal strong paired dominating set of G_1 or G_2 which is also a minimal strong paired dominating set of $G_1 + G_2$.

2. If G_1 has no isolated nodes and G_2 has an isolated node, then

$$\gamma_{spr}(G_1 + G_2) = \wedge \quad \{W(D), W(\{u, v\}): u \in V_1, v \in V_2\}$$

where D is any minimal strong paired dominating set of G_1 which is also a minimal strong paired dominating set of $G_1 + G_2$.

3. If G_1 has an isolated node and G_2 has no isolated nodes, then

$$\gamma_{spr}(G_1 + G_2) = \wedge \quad \{W(D), W(\{u, v\}): u \in V_1, v \in V_2\}$$

where D is any minimal strong paired dominating set of G_2 which is also a minimal strong paired dominating set of $G_1 + G_2$.

4. If both G_1 and G_2 have isolated nodes, then

$$\gamma_{spr}(G_1 + G_2) = \wedge \quad \{W(\{u, v\}): u \in V_1, v \in V_2\}$$

Proof: Note that for any two fuzzy graphs G_1 and G_2, its join $G_1 + G_2$ is a fuzzy graph having no isolated nodes and hence $\gamma_{spr}(G_1 + G_2)$ always exists. The proof of 1, 2, 3 and 4 of Theorem 21 is similar to that of Theorem 20.

Theorem 22. If G_1 and G_2 are two fuzzy fuzzy graphs on V_1 and V_2 respectively with $V_1 \cap V_2 = \phi$. Then

1. If both G_1 and G_2 are connected, then

$$\gamma_{sc}(G_1 + G_2) = \wedge \quad \{W(D), W(\{u, v\}): u \in V_1, v \in V_2\}$$

where D is any minimal strong connected dominating set of G_1 or G_2 which is also a minimal strong connected dominating set of $G_1 + G_2$.

2. If G_1 is connected and G_2 is not connected, then

$$\gamma_{sc}(G_1 + G_2) = \wedge \quad \{W(D), W(\{u, v\}): u \in V_1, v \in V_2\}$$

where D is any minimal strong connected dominating set of G_1 which is also a minimal strong connected dominating set of $G_1 + G_2$.

3.If G_1 is not connected and G_2 is connected, then

$$\gamma_{sc}(G_1 + G_2) = \wedge \quad \{W(D), W(\{u, v\}): u \in V_1, v \in V_2\}$$

where D is any minimal strong connected dominating set of G_2 which is also a minimal strong connetced dominating set of $G_1 + G_2$.

4. If both G_1 and G_2 are disconnected, then

$$\gamma_{sc}(G_1 + G_2) = \wedge \quad \{W(\{u, v\}): u \in V_1, v \in V_2\}$$

Proof: Note that for any two fuzzy graphs G_1 and G_2, the fuzzy graph $G_1 + G_2$ is connected and hence $\gamma_{sc}(G_1 + G_2)$ always exists. The proof of 1, 2, 3 and 4 of Theorem 22 is similar to that of Theorem 20.

Definition 23. [13, 14] Cartesian product of two fuzzy graphs: Let $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ be two fuzzy graphs with $G_1^*: (V_1, E_1)$ and $G_2^*: (V_2, E_2)$ with $V_1 \cap V_2 = \phi$. Consider the Cartesian product $G^* = G_1^* \times G_2^* = (V, E)$ where $V = V_1 \times V_2$ and $E = \{((u, u_2), (u, v_2)): u \in V_1, (u_2, v_2) \in E_2\} \cup \quad \{((u_1, w), (v_1, w)):, (u_1, v_1) \in E_1, w \in V_2\}$. Then the Cartesian product of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G = G_1 \times G_2: (\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$ defined by

$$(\sigma_1 \times \sigma_2)(u_1, u_2) = (\sigma_1(u_1) \wedge \quad \sigma_2)(u_2), \; \forall (u_1, u_2) \in V$$

and $(\mu_1 \times \mu_2)((u, u_2), (u, v_2)) = \sigma_1(u) \wedge \quad \mu_2(u_2, v_2) \; \forall u \in V_1, (u_2, v_2) \in E_2$

$(\mu_1 \times \mu_2)((u_1, w), (v_1, w)) = \sigma_2(w) \wedge \quad \mu_1(u_1, v_1) \; \forall w \in V_2, (u_1, v_1) \in E_1$

Definition 24. [13, 14] Composition of two fuzzy graphs: Let $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ be two fuzzy graphs with $G_1^*: (V_1, E_1)$ and $G_2^*: (V_2, E_2)$ with $V_1 \cap V_2 = \phi$. Consider the composition $G^* = G_1^* \circ G_2^* = (V, E)$ where $V = V_1 \times V_2$ and $E = \{((u, u_2), (u, v_2)): u \in V_1, (u_2, v_2) \in E_2\} \cup \quad \{((u_1, w), (v_1, w)):, (u_1, v_1) \in E_1, w \in V_2\} \cup \quad \{((u_1, u_2), (v_1, v_2)): (u_1, v_1) \in E_1, u_2 \neq v_2\}$. Then the composition of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G = G_1 \circ G_2: (\sigma_1 \circ \sigma_2, \mu_1 \circ \mu_2)$ defined by

$$(\sigma_1 \circ \sigma_2)(u_1, u_2) = (\sigma_1(u_1) \wedge \quad \sigma_2)(u_2), \; \forall (u_1, u_2) \in V$$

and $(\mu_1 \circ \mu_2)((u, u_2), (u, v_2)) = \sigma_1(u) \wedge \quad \mu_2(u_2, v_2) \; \forall u \in V_1, (u_2, v_2) \in E_2$

$(\mu_1 \circ \mu_2)((u_1, w), (v_1, w)) = \sigma_2(w) \wedge \quad \mu_1(u_1, v_1) \; \forall w \in V_2, (u_1, v_1) \in E_1$

$(\mu_1 \circ \mu_2)((u_1, u_2), (v_1, v_2)) = \sigma_2(u_2) \wedge \quad \sigma_2(v_2) \wedge \quad \mu_1(u_1, v_1) \quad \forall ((u_1, u_2), (v_1, v_2)) \in E - E^*$

where $E^* = \{((u, u_2), (u, v_2)): u \in V_1, (u_2, v_2) \in E_2\} \cup \quad \{((u_1, w), (v_1, w)): w \in V_2, (u_1, v_1) \in E_1\}$

Proposition 25. Let $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ be two fuzzy graphs with $G_1^*: (V_1, E_1)$ and $G_2^*: (V_2, E_2)$ with $V_1 \cap V_2 = \phi$. Then

$$CONN_{G_1 \circ G_2}((u_1, u_2), (v_1, v_2)) \leq$$

$\wedge \quad \{CONN_{G_1}(u_1, v_1), CONN_{G_2}(u_2, v_2)\}, \forall u_1, v_1 \in V_1, u_2, v_2 \in V_2$

Proof: By definition of connectivity of fuzzy graphs[14]

$$CONN_{G_1 \circ G_2}((u_1, u_2), (v_1, v_2)) = sup\{(\mu_1 \circ \mu_2)_k((u_1, u_2), (v_1, v_2)): k = 1, 2, 3 \cdots\}$$

Since $sup(f \circ g) \leq \wedge \{supf, supg\}$ the result follows.

Theorem 26. Let D_1 and D_2 be strong dominating sets of the fuzzy graphs $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ respectively. Then $D_1 \times D_2$ is a strong dominating set of $G_1 \circ G_2$.
Proof: We have to prove that for every arc $(u, v) \in (V_1 \times V_2) - (D_1 \times D_2)$ there exists an arc (u', v') such that $((u, v), (u', v')$ is a strong arc. Let $(u, v) \in (V_1 \times V_2) - (D_1 \times D_2)$. Then $u \notin D_1$ or $v \notin D_2$
CASE 1: $u \notin D_1$ and $v \in D_2$.
Since D_1 is a strong dominating set of G_1 there exists a node $u_1 \in D_1$ which strongly dominates u. That is, (u, u_1) is a strong arc. Hence $CONN_{G_1}(u, u_1) = \mu_1(u, u_1)$
Now, $(u_1, v) \in D_1 \times D_2$ and

$$(\mu_1 \circ \mu_2)((u, v), (u_1, v)) = \sigma_2(v) \wedge \mu_1(u, u_1)$$
$$= \sigma_2(v) \wedge CONN_{G_1}(u, u_1)$$
$$\geq CONN_{G_2}(v, v_i) \wedge CONN_{G_1}(u, u_1) \forall v_i \in V_2$$
$$\geq CONN_{G_1 \circ G_2}((u, v), (u_1, v))$$

Hence $((u, v), (u_1, v))$ is a strong arc. Hence (u_1, v) strongly dominates (u, v) in $G_1 \circ G_2$.
CASE II: $u \in D_1$ and $v \notin D_2$.
Since D_2 is a strong dominating set of G_2 there exists a node $v_2 \in D_2$ which strongly dominates v. That is, (v, v_2) is a strong arc. Hence $CONN_{G_2}(v, v_2) = \mu_2(v, v_2)$
Now, $(u, v_2) \in D_1 \times D_2$ and

$$(\mu_1 \circ \mu_2)((u, v), (u, v_2)) = \sigma_1(u) \wedge \mu_2(v, v_2)$$
$$= \sigma_1(u) \wedge CONN_{G_2}(v, v_2)$$
$$\geq CONN_{G_1}(u, u_i) \wedge CONN_{G_2}(v, v_2) \forall u_i \in V_1$$
$$\geq CONN_{G_1 \circ G_2}((u, v), (u, v_2))$$

Hence $((u, v), (u, v_2))$ is a strong arc. Hence (u, v_2) strongly dominates (u, v) in $G_1 \circ G_2$.
CASE III: $u \notin D_1$ and $v \notin D_2$.
Since D_1 and D_2 are strong dominating sets of G_1 and G_2 respectively, there exist nodes $u_1 \in D_1$ and $v_2 \in D_2$ such that u_1 strongly dominates u in G_1 and v_2 strongly dominates v in G_2.
Then $CONN_{G_1}(u, u_1) = \mu_1(u, u_1)$ and $CONN_{G_2}(v, v_2) = \mu_2(v, v_2)$
Now $(u_1, v_2) \in D_1 \times D_2$ and

$$(\mu_1 \circ \mu_2)((u, v), (u_1, v_2)) = \sigma_2(v) \wedge \sigma_2(v_2) \wedge \mu_1(u, u_1)$$
$$\geq \mu_2(v, v_2) \geq \wedge CONN_{G_1}(u, u_1)$$
$$= CONN_{G_2}(v, v_2) \wedge CONN_{G_1}(u, u_1)$$
$$\geq CONN_{G_1 \circ G_2}((u, v), (u_1, v_2))$$

Hence $((u, v), (u_1, v_2))$ is a strong arc. Hence (u_1, v_2) strongly dominates (u, v) in $G_1 \circ G_2$.
Hence $D_1 \times D_2$ is a strong dominating set of $G_1 \circ G_2$.

4. Conclusion

In this paper, the concept of domination is introduced on various operations on fuzzy graphs such as union,join, composition and cartesian product using strong arcs. The different types of strong domination parameters of union and join of two fuzzy graphs is obtained.It is obtained the strong domination set of cartesian product of two fuzzy graphs.

REFERENCES

1. K.R.Bhutani, On automorphisms of fuzzy graphs*, Pattern Recognition Letters,* 9 (1989) 159–162.
2. K.R.Bhutani and A.Rosenfeld, Strong arcs in fuzzy graphs, *Inform. Sci*., 152 (2003) 319–322.
3. K.R.Bhutani and A.Rosenfeld, Fuzzy end nodes in fuzzy graphs, *Inform. Sci*., 152 (2003) 323-326.
4. K.R.Bhutani and A.Batton, On M-strong fuzzy graphs, *Inform. Sci.,* 1559 (2003) 103–109.
5. G.Chartrand and P.Zang, Introduction to graph theory, *Tata McGraw-Hill Edition* 2006.
6. T.W.Haynes, S.T.Hedetniemi and P.J.Slater, Fundamentals of domination in graphs, *Marcel Dekker, Inc., New York* 1998.
7. O.T.Manjusha and M.S.Sunitha, Notes on domination in fuzzy graphs, *Journal of intelligent and Fuzzy Systems,* 27 (2014) 3205–3212.
8. O.T.Manjusha and M.S.Sunitha, Total domination in fuzzy graphs using strong arcs, *Annals of Pure and Applied Mathematics,* 9(1) (2014) 23–33.
9. O.T.Manjusha and M.S.Sunitha, Strong domination in fuzzy graphs, *Fuzzy Inf. Eng.,* 7 (2015) 369-377.
10. O.T.Manjusha and M.S.Sunitha, Connected domination in fuzzy graphs using strong arcs, *Annals of Fuzzy Mathematics and Informatics*, Accepted.
11. O.T.Manjusha and M.S.Sunitha, Independent domination in fuzzy graphs using strong arcs,, Communicated.
12. O.T.Manjusha and M.S.Sunitha, Coverings, matchings and paired domination in fuzzy graphs using strong arcs*, Iranian Journal of Fuzzy Systems,* 16(1) (2019) 145-157
13. J.N.Mordeson and P.C.-Shyh, Operations on fuzzy graphs*, Inform. Sci.,* 79 (1994) 159–170.
14. J.N.Mordeson and P.S.Nair, *Fuzzy graphs and fuzzy hypergraphs*, *Physica - Verlag* 2000.
15. A.Nagoorgani and V.T.Chandrasekaran, Domination in fuzzy graph, *Adv. in Fuzzy Sets and Systems,* 1(1) (2006) 17–26.
16. A.Nagoorgani and P.Vijayalakshmi, Insensitive arc in domination of fuzzy graph, *Int. J. Contemp. Math. Sciences,* 6(26) (2011) 1303–1309.
17. A.Nagoor Gani and P.Vadivel, A study on domination, independent domination and irredundance in fuzzy graph, *Applied Mathematical Sciences*, 5(47) (2011) 2317 - 2325.
18. P. Debnath, Domination in interval-valued fuzzy graphs, *Ann. Fuzzy Math. Inform.,*

6(2) (2013) 363-370.

19. A.Rosenfeld, *Fuzzy graphs, in fuzzy sets and their application to cognitive and decision processes,* Academic Press, (1975) 77–95.

20. E.Sampathkumar and H.B.Walikar, The connected domination number of a graph, *Jour. Math. Phy. Sci.,* 13(6) (1979) 607–613.

21. K.R.Sandeep Narayan and M.S.Sunitha, Connectivity in a fuzzy graph and its complement, *Gen. Math. Notes,* 9(1) (2012) 38–43.

22. A.Somasundaram and S.Somasundaram, Domination in fuzzy graphs-I, *Pattern Recognition Letters,* 19 (1998) 787–791.

23. A.Somasundaram, Domination in fuzzy graphs-II, *J. Fuzzy Math*., 13 (2) (2005) 281–288.

24. A.Somasundaram, Domination in products of fuzzy graphs, *Inter. Journal of Uncertainty Fuzziness and Knowledge-Based Systems,* 2 (2005) 195-204.

25. Sunil Mathew and M.S.Sunitha, Types of arcs in a fuzzy graph, *Inform. Sci.,* 179 (2009) 1760–1768.

26. S. Mathew and M.S.Sunitha, Node connectivity and arc connectivity of a fuzzy graph, *Inform. Sci.,* 180 (2010) 519–531.

27. M.S.Sunitha and A.Vijayakumar, Complement of a fuzzy graph, *Indian Journal of Pure and Applied Mathematics,* 33(9) (2002) 1451–1464.

28. M.S.Sunitha and A.Vijayakumar, A characterization of fuzzy trees, *Inform. Sci.,* 113 (1999) 293–300.

29. A.A.Talebi and H.Rashmanlou, Isomorphism on interval valued fuzzy graphs, *Ann. Fuzzy Math. Inform,* 6(1) (2013) 47–58.

30. R.T.Yeh and S.Y.Bang, Fuzzy relations, fuzzy graphs and their applications to clustering analysis, in: L.A. Zadeh, K.S. Fu, M. Shimura (Eds.), Fuzzy sets and Their Applications, *Academic Press* (1975) 125–149.

31. M.Cary, Perfectly regular and perfectly edge-regular fuzzy graphs, *Annals of Pure and Applied Mathematics,* 16(2) (2018) 461-469.

32. S.Mathew and M.S.Sunitha, Fuzzy graph theory: a survey, *Annals of Pure and Applied Mathematics,* 4(1) (2013) 92-110.

33. T.Pathinadhan and J.Jesintha, Matrix representation of double layered fuzzy graph its properties, *Annals of Pure and Applied Mathematics,* 8(2) (2014) 51-58.