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On the Class of the Diophantine Equations $(10K + A)^x + (10M + A)^y = z^2$ when $A = 1, 3, 7, 9$ with Positive Integers x, y, z

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Abstract. In this article we consider the class of Diophantine equations $(10K + A)^x +$ $(10M + A)^y = z^2$ when $A = 1, 3, 7, 9$ with positive integers x, y, z. It is established: (i) When $A = 1$, the equation has no solutions. (ii) When $A = 3, 7, 9$, each equation has infinitely many solutions. All the results are achieved by our recent new technique which makes use of the last digits of the powers involved. Various solutions for the equations are exhibited.

Keywords: Diophantine equations

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds. Among them are for example [7, 8, 9, 10, 11].

In this article we consider the class of Diophantine equations $(10K + A)^x + (10M + A)^y = z^2$ when $A = 1, 3, 7, 9$ with positive integers x, y, z. We will show for $A = 1$ that the equation has no solutions, and when $A = 3, 7, 9$, that each equation has infinitely many solutions. All these results are attained by using the last digits of the powers involved. This is a new elementary technique developed by us for finding solutions of exponential Diophantine equations already adopted in our recent articles on this subject. The solutions obtained correspond to primes and composites as well with no distinction. Each of the following sections is self-contained, and so are the theorems within each section.

2. On $(10K + 1)^x + (10M + 1)^y = z^2$

In a very short, elementary and elegant way, we will show in the following theorem that the equation $(10K + 1)^x + (10M + 1)^y = z^2$ has no solutions.

Theorem 2.1. Let $K \geq 1$ and $M \geq 1$ be integers. Let x, y, z be positive integers. Then

$$(10K + 1)^x + (10M + 1)^y = z^2 \tag{1}$$

has no solutions.

Proof: For all values K, x, M, y, each of the two powers $(10K + 1)^x$ and $(10M + 1)^y$ has a last digit which is equal to 1. Thus, the sum $(10K + 1)^x + (10M + 1)^y$ has a last digit which is equal to 2. If the sum satisfies (1), then z^2 is even. Any even square z^2 does not have a last digit which is equal to 2. Therefore, a priori (1) has no solutions.

The proof of Theorem 2.1 is complete. □

Remark 2.1. In Theorem 2.1 we have established for all integers (composites, primes) whose last digit ends in 1, and for all values $x \geq 1$, $y \geq 1$ that equation (1) has no solutions. In particular, when $10K + 1 = p$, $10M + 1 = q$ are distinct primes, then the equation $p^x + q^y = z^2$ has no solutions. The result is also valid when K, M are equal. Thus, it is completely redundant to consider any equations of the form (1) since such equations have no solutions.

3. On $(10K + 3)^x + (10M + 3)^y = z^2$

Let $K \geq 0$, $x \geq 1$, $M \geq 0$, $y \geq 1$ be integers. Let z be a positive integer. For all values x, y, each of the powers $(10K + 3)^x$, $(10M + 3)^y$ ends in one of the digits 3, 9, 7, 1. Suppose that for some values K, x, M, y, z

$$(10K + 3)^x + (10M + 3)^y = z^2 \tag{2}$$

is satisfied. The sum in (2) is even, and ends in one of the digits 2, 4, 6, 8, 0. Since z^2 is an even square, z^2 cannot have a last digit equal to 2 or equal to 8. Hence z^2 ends in one of the digits 4, 6, 0. To prove that (2) has infinitely many solutions, it is clearly immaterial what three possibilities for z^2 are chosen. A set of three such possibilities is presented in the following Table 1.

Table 1.

case	**last digit of $(10K + 3)^x$**	**last digit of $(10M + 3)^y$**	**last digit of $(10K + 3)^x + (10M + 3)^y$**	**solutions of $(10K + 3)^x + (10M + 3)^y = z^2$**
1	3	1	4	infinitely many
2	3	3	6	infinitely many
3	3	7	0	infinitely many

In Theorems 3.1 – 3.3 we shall consider the three cases in Table 1. We will show that in each such case, equation (2) has infinitely many solutions.

Theorem 3.1. If $(10K + 3)^x$ has a last digit equal to 3, and $(10M + 3)^y$ has a last digit equal to 1, then the equation $(10K + 3)^x + (10M + 3)^y = z^2$ has infinitely many solutions.

Proof: When $(10K + 3)^x$ ends in the digit 3, then $x = 4m + 1$ where $m \geq 0$ is an integer. When $(10M + 3)^y$ ends in the digit 1, then $y = 4n$ where $n \geq 1$ is an integer. To prove our assertion, it suffices to consider x, y as the smallest possible fixed values. Let $x = 1$ $(m = 0)$ and $y = 4$ $(n = 1)$ be fixed values. Set $z = (10M + 3)^2 + 9$ valid for each value $M \geq 0$. Then z ends in 8, and z^2 has a last digit equal to 4 as in Table 1. For all values $M \geq 0$ let $K = 180M^2 + 108M + 24$. Then for $(10K + 3)^1 + (10M + 3)^4 = z^2$ we have the identity

$$(10(180M^2 + 108M + 24) + 3)^1 + (10M + 3)^4 = ((10M + 3)^2 + 9)^2 \qquad (3)$$

valid for each and every value $M \geq 0$.

Thus, the equation $(10K + 3)^1 + (10M + 3)^4 = z^2$ has infinitely many solutions.

This completes the proof of Theorem 3.1. □

The first three solutions obtained from (3) are:

Solution 1.	$243^1 + 3^4 = 18^2$	$M = 0$,	$K = 24$,	$x = 1$,	$y = 4$.
Solution 2.	$3123^1 + 13^4 = 178^2$	$M = 1$,	$K = 312$,	$x = 1$,	$y = 4$.
Solution 3.	$9603^1 + 23^4 = 538^2$	$M = 2$,	$K = 960$,	$x = 1$,	$y = 4$.

Remark 3.1. Suppose that $x = 5$ $(m = 1)$ and $y = 4$ $(n = 1)$ are fixed values. When $K = M$, we have

$$(10K + 3)^5 + (10K + 3)^4 = (10K + 3)^4\big((10K + 3) + 1\big) = \big((10K + 3)^2\big)^2 (10K + 4) = z^2$$

provided $(10K + 4)$ is a square. The first four values K for which $(10K + 4)$ is a square are $K = 0, 6, 14, 32, \ldots,$ and so on. Infinitely many such values K exist for which $(10K + 4)$ is a square. The first four solutions are then:

$3^5 + 3^4 = 18^2$, $63^5 + 63^4 = 31752^2$, $143^5 + 143^4 = 245388^2$, $323^5 + 323^4 = 1877922^2$.

It follows that the values z alternate between z ending in 8 and z ending in 2. The value z^2 clearly has a last digit equal to 4 as in Table 1.

Theorem 3.2. If $(10K + 3)^x$ has a last digit equal to 3, and $(10M + 3)^y$ has a last digit equal to 3, then the equation $(10K + 3)^x + (10M + 3)^y = z^2$ has infinitely many solutions.

Proof: When $(10K + 3)^x$ ends in the digit 3, then $x = 4m + 1$ where $m \geq 0$ is an integer. When $(10M + 3)^y$ ends in the digit 3, then $y = 4n + 1$ where $n \geq 0$ is an integer. To prove our assertion, it suffices to consider x, y as the smallest possible fixed values. Let $x = 1$ $(m = 0)$ and $y = 1$ $(n = 0)$ be fixed values. Set $z = (10M + 3) + 1$ valid for each value for $M \geq 0$. Then z ends in 4, and z^2 has a last digit equal to 6 as in Table 1. For all values $M \geq 0$, let $K = 10M^2 + 7M + 1$. We then obtain for $(10K + 3)^1 + (10M + 3)^1 = z^2$ the identity

$$\big(10(10M^2 + 7M + 1) + 3\big)^1 + (10M + 3)^1 = \big((10M + 3) + 1\big)^2 \qquad (4)$$

valid for each and every value $M \geq 0$.

The equation $(10K + 3)^1 + (10M + 3)^1 = z^2$ has infinitely many solutions.

This concludes the proof of Theorem 3.2. □

The first three solutions which follow from (4) are:

Solution 4.	$13^1 + 3^1 = 4^2$	$M = 0$,	$K = 1$,	$x = y = 1$.
Solution 5.	$183^1 + 13^1 = 14^2$	$M = 1$,	$K = 18$,	$x = y = 1$.
Solution 6.	$553^1 + 23^1 = 24^2$	$M = 2$,	$K = 55$,	$x = y = 1$.

Theorem 3.3. If $(10K + 3)^x$ has a last digit equal to 3, and $(10M + 3)^y$ has a last digit equal to 7, then the equation $(10K + 3)^x + (10M + 3)^y = z^2$ has infinitely many solutions.

Proof: When $(10K + 3)^x$ ends in the digit 3, then $x = 4m + 1$ where $m \geq 0$ is an integer. When $(10M + 3)^y$ ends in the digit 7, then $y = 4n + 3$ where $n \geq 0$ is an integer. To prove our assertion, it suffices to consider x, y as the smallest possible fixed values. Let $x = 1$ $(m = 0)$ and $y = 3$ $(n = 0)$ be fixed values. Set $z = (10M + 3)^2 + 1$ valid for each value for $M \geq 0$. Then z and also z^2 have a last digit equal to 0 as in Table 1. For all values $M \geq 0$ let $K = 1000M^4 + 1100M^3 + 470M^2 + 93M + 7$. We then obtain for $(10K + 3)^1 + (10M + 3)^3 = z^2$ the identity

$$(10(1000M^4 + 1100M^3 + 470M^2 + 93M + 7) + 3)^1 + (10M + 3)^3 = ((10M + 3)^2 + 1)^2 \qquad (5)$$

valid for each and every value $M \geq 0$.

The equation $(10K + 3)^1 + (10M + 3)^3 = z^2$ has infinitely many solutions.

This completes the proof of Theorem 3.3. □

The first four solutions derived from (5) are:

Solution 7.	$73^1 + 3^3 = 10^2$	$M = 0$,	$K = 7$,	$x = 1$,	$y = 3$.
Solution 8.	$26703^1 + 13^3 = 170^2$	$M = 1$,	$K = 2670$,	$x = 1$,	$y = 3$.
Solution 9.	$268733^1 + 23^3 = 530^2$	$M = 2$,	$K = 26873$,	$x = 1$,	$y = 3$.
Solution 10.	$1152163^1 + 33^3 = 1090^2$	$M = 3$,	$K = 115216$,	$x = 1$,	$y = 3$.

4. On $(10K + 7)^x + (10M + 7)^y = z^2$

Let $K \geq 0$, $x \geq 1$, $M \geq 0$, $y \geq 1$ be integers. Let z be a positive integer. For all values x, y, each of the powers $(10K + 7)^x$, $(10M + 7)^y$ ends in one of the digits 7, 9, 3, 1. Suppose that for some values K, x, M, y, z

$$(10K + 7)^x + (10M + 7)^y = z^2 \qquad (6)$$

is satisfied. The sum in (6) is even, and ends in one of the digits 2, 4, 6, 8, 0. Since z^2 is even, z^2 cannot have a last digit equal to 2 or equal to 8. Thus z^2 ends in one of the digits 4, 6, 0. To prove that (6) has infinitely many solutions, it is clearly immaterial what three possibilities for z^2 are chosen. A set of three possibilities is demonstrated in the following Table 2.

Table 2.

Case	**last digit of $(10K + 7)^x$**	**last digit of $(10M + 7)^y$**	**last digit of $(10K + 7)^x + (10M + 7)^y$**	**solutions of $(10K + 7)^x + (10M + 7)^y = z^2$**
1	7	7	4	infinitely many
2	7	9	6	infinitely many
3	7	3	0	infinitely many

In the following Theorems 4.1 – 4.3 we consider the three cases in Table 2. We will show that in each case, equation (6) has infinitely many solutions.

Theorem 4.1. If $(10K + 7)^x$ has a last digit equal to 7, and $(10M + 7)^y$ has a last digit equal to 7, then the equation $(10K + 7)^x + (10M + 7)^y = z^2$ has infinitely many solutions.

Proof: When $(10K + 7)^x$ ends in the digit 7, then $x = 4m + 1$ where $m \ge 0$ is an integer. When $(10M + 7)^y$ ends in the digit 7, then $y = 4n + 1$ where $n \ge 0$ is an integer. To prove our assertion, it suffices to consider x, y as the smallest possible fixed values. Let $x = 1$ $(m = 0)$ and $y = 1$ $(n = 0)$ be fixed values. Set $z = (10M + 7) + 1$ valid for each value $M \ge 0$, and z^2 has a last digit equal to 4 as in Table 2. For all values $M \ge 0$ let $K = 10M^2 + 15M + 5$. Then for $(10K + 7)^1 + (10M + 7)^1 = z^2$ we obtain the identity

$$\big(10(10M^2 + 15M + 5) + 7\big)^1 + (10M + 7)^1 = \big((10M + 7) + 1\big)^2 \qquad (7)$$

valid for each and every value $M \ge 0$.

The equation $(10K + 7)^1 + (10M + 7)^1 = z^2$ has infinitely many solutions.

The proof of Theorem 4.1 is complete. □

The first three solutions obtained from (7) are:

Solution 11.	$57^1 + 7^1 = 8^2$	$M = 0,$	$K = 5,$	$x = y = 1.$
Solution 12.	$307^1 + 17^1 = 18^2$	$M = 1,$	$K = 30,$	$x = y = 1.$
Solution 13.	$757^1 + 27^1 = 28^2$	$M = 2,$	$K = 75,$	$x = y = 1.$

Theorem 4.2. If $(10K + 7)^x$ has a last digit equal to 7, and $(10M + 7)^y$ has a last digit equal to 9, then the equation $(10K + 7)^x + (10M + 7)^y = z^2$ has infinitely many solutions.

Proof: When $(10K + 7)^x$ ends in the digit 7, then $x = 4m + 1$ where $m \geq 0$ is an integer. When $(10M + 7)^y$ ends in the digit 9, then $y = 4n + 2$ where $n \geq 0$ is an integer. To prove our assertion, it suffices to consider x, y as the smallest possible fixed values. Let $x = 1$ $(m = 0)$ and $y = 2$ $(n = 0)$ be fixed values. Set $z = (10M + 7) + 9$ valid for each value $M \geq 0$. Then z and z^2 have a last digit equal to 6 as in Table 2. For all values $M \geq 0$ let $K = 18M + 20$. Then for $(10K + 7)^1 + (10M + 7)^2 = z^2$ we obtain the identity

$$\big(10(18M + 20) + 7\big)^1 + (10M + 7)^2 = \big((10M + 7) + 9\big)^2 \tag{8}$$

valid for each and every value $M \geq 0$.

The equation $(10K + 7)^1 + (10M + 7)^2 = z^2$ has infinitely many solutions.

This concludes the proof of Theorem 4.2. □

The following three solutions stem from (8).

Solution 14.	$207^1 + 7^2 = 16^2$	$M = 0$,	$K = 20$,	$x = 1$,	$y = 2$.
Solution 15.	$387^1 + 17^2 = 26^2$	$M = 1$,	$K = 38$,	$x = 1$,	$y = 2$.
Solution 16.	$567^1 + 27^2 = 36^2$	$M = 2$,	$K = 56$,	$x = 1$,	$y = 2$.

Theorem 4.3. If $(10K + 7)^x$ has a last digit equal to 7, and $(10M + 7)^y$ has a last digit equal to 3, then the equation $(10K + 7)^x + (10M + 7)^y = z^2$ has infinitely many solutions.

Proof: When $(10K + 7)^x$ ends in the digit 7, then $x = 4m + 1$ where $m \geq 0$ is an integer. When $(10M + 7)^y$ ends in the digit 3, then $y = 4n + 3$ where $n \geq 0$ is an integer. To prove our assertion, it suffices to consider x, y as the smallest possible fixed values. Let $x = 1$ $(m = 0)$ and $y = 3$ $(n = 0)$ be fixed values. Set $z = (10M + 7)^2 + 1$ valid for each value $M \geq 0$. Then z and also z^2 have a last digit equal to 0 as in Table 2. For all values $M \geq 0$ let $K = 1000M^4 + 2700M^3 + 2750M^2 + 1253M + 215$. We then obtain for $(10K + 7)^1 + (10M + 7)^3 = z^2$ the identity

$$(10(1000M^4 + 2700M^3 + 2750M^2 + 1253M + 215) + 7)^1 + (10M + 7)^3 = ((10M + 7)^2 + 1)^2 \tag{9}$$

valid for each and every value $M \geq 0$.

The equation $(10K + 7)^1 + (10M + 7)^3 = z^2$ has infinitely many solutions.

The proof of Theorem 4.3 is complete. □

The first three solutions which follow from (9) are:

Solution 17.	$2157^1 + 7^3 = 50^2$	$M = 0$,	$K = 215$,	$x = 1$,	$y = 3$.
Solution 18.	$79187^1 + 17^3 = 90^2$	$M = 1$,	$K = 7918$,	$x = 1$,	$y = 3$.
Solution 19.	$513217^1 + 27^3 = 730^2$	$M = 2$,	$K = 51321$,	$x = 1$,	$y = 3$.

5. On $(10K + 9)^x + (10M + 9)^y = z^2$

Let $K \geq 0$, $x \geq 1$, $M \geq 0$, $y \geq 1$ be integers. Let z be a positive integer. For all values x, y, each of the powers $(10K + 9)^x$, $(10M + 9)^y$ ends either in the digit 1 or in the digit 9. Suppose that for some values K, x, M, y, z

$$(10K + 9)^x + (10M + 9)^y = z^2 \tag{10}$$

is satisfied. The sum in (10) is even, and therefore ends in one of the digits 0, 2, 8. From (10) it follows that z^2 is even, and as such cannot have a last digit equal to 2 or equal to 8. Hence, z^2 must end in the digit 0. For our purposes, it is clearly immaterial whether $(10K + 9)^x$ ends in 9 and $(10M + 9)^y$ ends in 1 or vice versa. Without any loss of generality we shall consider the case when $(10K + 9)^x$ ends in 9, and $(10M + 9)^y$ ends in 1.

Theorem 5.1. If $(10K + 9)^x$ has a last digit equal to 9, and $(10M + 9)^y$ has a last digit equal to 1, then the equation $(10K + 9)^x + (10M + 9)^y = z^2$ has infinitely many solutions.

Proof: When $(10K + 9)^x$ ends in the digit 9, then $x = 2m + 1$ where $m \geq 0$ is an integer. When $(10M + 9)^y$ ends in the digit 1, then $y = 2n$ where $n \geq 1$ is an integer. The value z^2 has a last digit equal to 0, and so does the value z.

To prove the infinitude of solutions, it suffices to consider x, y as the smallest possible fixed values. Let $x = 1$ $(m = 0)$ and $y = 2$ $(n = 1)$ be fixed values. Set $z = (10M + 9) + 1$ which has a last digit equal to 0. For all values $M \geq 0$ let $K = 2M + 1$. We then obtain the identity

$$\big(2(10M + 9) + 1\big)^1 + (10M + 9)^2 = \big((10M + 9) + 1\big)^2 \tag{11}$$

valid for each and every value $M \geq 0$.

The equation $(10K + 9)^1 + (10M + 9)^2 = z^2$ has infinitely many solutions.

This concludes the proof of Theorem 5.1. □

We now demonstrate the first five solutions obtained from (11).

Solution 20.	$19^1 + 9^2 = 10^2$	$K = 1$,	$x = 1$,	$M = 0$,	$y = 2$.
Solution 21.	$39^1 + 19^2 = 20^2$	$K = 3$,	$x = 1$,	$M = 1$,	$y = 2$.
Solution 22.	$59^1 + 29^2 = 30^2$	$K = 5$,	$x = 1$,	$M = 2$,	$y = 2$.
Solution 23.	$79^1 + 39^2 = 40^2$	$K = 7$,	$x = 1$,	$M = 3$,	$y = 2$.
Solution 24.	$99^1 + 49^2 = 50^2$	$K = 9$,	$x = 1$,	$M = 4$,	$y = 2$.

Remark 5.1. The values $10K + 9$ and $10M + 9$ yield primes and composites. The five solutions above, clearly show that the solutions of $(10K + 9)^x + (10M + 9)^y = z^2$ are composed of primes and composites (**solutions 20**, **21**, **23**), of primes only (**solution**

22), and of composites only (**solution 24**). Certainly, there exist infinitely many solutions of each category.

Remark 5.2. Let x, y be some fixed values. We observe that when z is a particular fixed value, then $(10K + 9)^x + (10M + 9)^y = z^2$ has more than one solution for z. This is shown for instance for $z = 40$ in the following three solutions.

Solution 25. $1519^1 + 9^2 = 40^2$.
Solution 26. $1239^1 + 19^2 = 40^2$.
Solution 27. $759^1 + 29^2 = 40^2$.

Thus, for fixed values $x = 1$, $y = 2$ and $z = 40$, it follows that the equation $(10K + 9)^x + (10M + 9)^y = z^2$ has exactly four solutions, namely **solution 23** and **solutions 25**, **26**, **27**.

Final remark. It has been shown in this article that the equation $(10K + 1)^x + (10M + 1)^y = z^2$ has no solutions, whereas the equations $(10K + 3)^x + (10M + 3)^y = z^2$, $(10K + 7)^x + (10M + 7)^y = z^2$ and $(10K + 9)^x + (10M + 9)^y = z^2$ have infinitely many solutions. The results for the three equations were achieved via identities. All the results in this paper stem from the use of the last digits of the given powers. Our technique although quite elementary, but rather very powerful, has already established itself in previous articles [see 1, 2, 3, 4, 5, 6] on exponential Diophantine equations of the form $p^x + q^y = z^2$. We believe that more equations of the kind may be solved in this manner.

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