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All the Solutions of the Diophantine Equation $p^x + p^y = z^4$ when $p \geq 2$ is Prime and x, y, z are Positive Integers

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Abstract. In this paper we consider the equation $p^x + p^y = z^4$ when x, y, z are positive integers, and establish the following results. (i) For $p = 2$ with equal values x, y the equation has infinitely many solutions, whereas when x, y are distinct values no solutions exist. (ii) For all primes $p > 2$, the equation has no solutions.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds.

In this paper, we consider the equation $p^x + p^y = z^4$ when $p \geq 2$ is prime and x, y, z are positive integers. When $p = 2$ with equal values x, y, it is shown that the equation has infinitely many solutions. Whereas, when $p = 2$ and x, y are distinct values, the equation has no solutions. Furthermore, for all primes $p > 2$ it is established that the equation has no solutions. This is done in the following two self-contained theorems.

2. All the solutions of $p^x + p^y = z^4$ when $p \geq 2$ is prime

When x, y, z are positive integers, we shall consider for the equation $p^x + p^y = z^4$ two cases, namely $p = 2$ and $p \geq 3$. The results will be demonstrated in the following Theorem 2.1 and Theorem 2.2.

Theorem 2.1. Suppose that $p = 2$.
(a) When $x = y$, then the equation $2^x + 2^y = z^4$ has infinitely many solutions.

(b) When x, y are distinct, then the equation $2^x + 2^y = z^4$ has no solutions.

Proof: (a) Let $x = y$. The equation

$$2^x + 2^y = 2 \cdot 2^x = z^4 \tag{1}$$

clearly has solutions only if $x = 4n + 3$ where $n \geq 0$ is an integer. Equation (1) then results in

$$2^{4n+3} + 2^{4n+3} = 2 \cdot 2^{4n+3} = 2^{4n+4} = (2^{n+1})^4 = z^4$$

an identity valid for each and every value $n \geq 0$. For all values n, the solutions of equation (1) are given by

$$(p, x, y, z) = (2, 4n + 3, 4n + 3, 2^{n+1}). \tag{2}$$

Thus, when $x = y$, the equation $2^x + 2^y = z^4$ has infinitely many solutions as asserted.

The proof of **(a)** is complete.

(b) Suppose that x, y are distinct. Without loss of generality let $x < y$. We shall assume that $2^x + 2^y = z^4$ has a solution and reach a contradiction. We have the equation

$$2^x + 2^y = 2^x(2^{y-x} + 1) = z^4. \tag{3}$$

If x is odd in (3), then (3) is clearly impossible. Hence, by our assumption x must be even, and the only possibility is $x = 4n$ where $n \geq 1$ is an integer. Then $2^x = 2^{4n} = (2^n)^4$, and by our assumption $2^{y-x} + 1$ must therefore equal K^4 where K is an odd integer. We have $2^{y-x} + 1 = K^4$ or

$$2^{y-x} = K^4 - 1 = K^4 - 1^4 = (K^2 - 1^2)(K^2 + 1^2) = (K - 1)(K + 1)(K^2 + 1^2). \tag{4}$$

In (4), the value $K = 1$ is impossible. Thus $K \geq 3$. If (4) is satisfied for some value K, then the three even factors $(K - 1)$, $(K + 1)$ and $(K^2 + 1)$ must simultaneously be equal to three distinct powers of 2. The factors $(K - 1)$ and $(K + 1)$ differ by 2 which is the smallest possible difference for two distinct powers of 2. The difference 2 is achieved only when $K - 1 = 2^1$ and $K + 1 = 2^2$. Thus $K = 3$ is uniquely determined. But, when $K = 3$, the factor $K^2 + 1 = 10$ is a multiple of 5 and (4) is impossible. This implies that the even factors $(K - 1)$, $(K + 1)$ and $(K^2 + 1)$ are never powers of 2 simultaneously. Hence, for all odd values $K \geq 3$, it follows that $2^{y-x} + 1 \neq K^4$, a contradiction. Our assumption is therefore false, and when x, y are distinct integers the equation $2^x + 2^y = z^4$ has no solutions.

This concludes part **(b)** and the proof of Theorem 2.1. □

Theorem 2.2. Suppose that $p \geq 3$ is prime.
(c) When $x = y$, the equation $p^x + p^y = z^4$ has no solutions.
(d) When x, y are distinct, the equation $p^x + p^y = z^4$ has no solutions.

Proof: We shall assume that $p^x + p^y$ has a solution and reach a contradiction.
(c) Let $x = y$. Then we have the equation

$$p^x + p^y = 2 \cdot p^x = z^4. \tag{5}$$

Since p is odd, equation (5) is impossible. Thus $x \neq y$.

(d) Suppose that x, y are distinct. Without loss of generality let $x < y$. We obtain

$$p^x + p^y = p^x(p^{y-x} + 1) = z^4. \qquad (6)$$

In (6), one can clearly see that x cannot assume any odd value. Therefore, by our assumption x is even. The only possibility then is $x = 4m$ where $m \geq 1$ is an integer.

Thus $p^x = p^{4m} = (p^m)^4$, and by our assumption $p^{y-x} + 1$ is equal to M^4 where M is an even integer. We have $p^{y-x} + 1 = M^4$ or

$$p^{y-x} = M^4 - 1 = M^4 - 1^4 = (M^2 - 1^2)(M^2 + 1^2) = (M - 1)(M + 1)(M^2 + 1^2). \qquad (7)$$

Let $p \geq 3$ be any fixed prime. If $M = 2$, then for all primes $p \geq 3$ in (7), $p^{y-x} \neq 1 \cdot 3 \cdot 5$. Hence $M \neq 2$, and $M \geq 4$. The three odd factors $(M - 1)$, $(M + 1)$ and $(M^2 + 1)$ in (7) must simultaneously be equal to three distinct powers of p if (7) is satisfied. The factors $(M - 1)$ and $(M + 1)$ differ by 2. Hence, when $p \mid (M - 1)$, then $p \nmid (M + 1)$ since $p \geq 3$. The impossibility of (7) then follows, and the contradiction is derived. Therefore, for all even values M, $p^{y-x} + 1 \neq M^4$. When x, y are distinct, then $p^x + p^y \neq z^4$ and our assumption is false.

This concludes part **(d)** and the proof of Theorem 2.2. □

3. Conclusion

When $p = 2$, we have established for the equation $p^x + p^y = z^4$ the identity $2^{4n+3} + 2^{4n+3} = (2^{n+1})^4$ valid for each and every value $n \geq 0$. Thus, the equation has infinitely many solutions all of which are presented in (2), and each such solution is unique. For all primes $p > 2$ the equation has no solutions.

In [1], we have considered the equation $p^x + p^y = z^2$ in which the current power z^4 is equal to the power z^2. We have established for $p = 3$ the infinite set of solutions

$$(p,\ x,\ y,\ z) = (3,\ 2t + 1,\ 2t,\ 2 \cdot 3^t) \qquad \text{for all integers } t \geq 1.$$

For all primes $p > 3$, it has been shown that the equation has no solutions.

Moreover, for $p = 2$ with $x = y$ and with $x > y$, two infinite sets of solutions have been achieved, namely:

$$(2,\ x,\ y,\ z) = (2,\ 2t + 1,\ 2t + 1,\ 2^{t+1}) \qquad \text{for all integers } t \geq 1,$$
$$(2,\ x, y, z) = (2,\ 2t + 3,\ 2t,\ 3 \cdot 2^t) \qquad \text{for all integers } t \geq 1.$$

For all other values x, y, z, it has been shown that the equation has no solutions.

Evidently, when no conditions are imposed on z such as z is also a square, then more solutions to the equation $p^x + p^y = z^2$ are achieved.

We remark that to the best of our knowledge, other authors have not considered equations of the form $p^x + q^y = z^4$. It is therefore obvious, that there are no references on such an equation.

REFERENCES

1. N. Burshtein, All the solutions of the diophantine equations $p^x + p^y = z^2$ and $p^x - p^y = z^2$ when $p \geq 2$ is prime, *Annals of Pure and Applied Mathematics*, 19 (2) (2019) 111-119.
2. N. Burshtein, The diophantine equation $x^4 + y^2 = z^4$ is insolvable in positive integers when at least one of x, y is prime, *Annals of Pure and Applied Mathematics*, 18 (2) (2018) 181-184.
3. N. Burshtein, On solutions to the diophantine equation $p^2 + q^2 = z^4$, *Annals of Pure and Applied Mathematics*, 18 (2) (2018) 135-138.
4. N. Burshtein, The class of diophantine equations $p^4 + q^y = z^4$ when $y = 1, 2, 3$ is insolvable for all primes p and q, *Annals of Pure and Applied Mathematics*, 17 (2) (2018) 193-195.
5. N. Burshtein, A note on the diophantine equation $p^3 + q^2 = z^4$ when p is prime, *Annals of Pure and Applied Mathematics*, 14 (3) (2017) 509-511.
6. N. Burshtein, On solutions to the diophantine equation $p^x + q^y = z^4$, *Annals of Pure and Applied Mathematics*, 14 (1) (2017) 63-68.
7. S. Chotchaisthit, On the diophantine equation $4^x + p^y = z^2$ where p is a prime number, *Amer. J. Math. Sci*., 1 (1) (2012) 191-193.
8. B. Poonen, Some diophantine equations of the form $x^n + y^n = z^m$, *Acta Arith*., 86 (1998) 193-205.
9. B. Sroysang, More on the diophantine equation $8^x + 19^y = z^2$, *Int. J. Pure Appl. Math*., 81 (4) (2012) 601-604.
10. A. Suvarnamani, Solution of the diophantine equation $p^x + q^y = z^2$, *Int. J. Pure Appl. Math*., 94 (4) (2014) 457-460.