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On the Diophantine Equations $3^x + 6^y = z^2$ and $5^x + 8^y = z^2$

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Abstract. In this article, we show that the Diophantine equation $3^x + 6^y = z^2$ has a finite number of solutions whereas the equation $5^x + 8^y = z^2$ has no solutions in positive integers x, y, z.

Keywords: Exponential Diophantine equations

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

A Diophantine equation is a polynomial equation that takes only integer values. There are various forms of diophantine equations studied by different mathematicians [5-7] in the last couple of decades.

The famous general equation $p^x + q^y = z^2$ has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds. In 2011, Suvarnamani [8] proved that the solution to the diophantine equation of the form $2^x + p^y = z^2$ is $(x, y, z) = (2k, 1, 1 + 2k)$ if $p = 1 + 2^{k+1}$.Variousauthors [2-4] have investigated equations of the form $p^x + (p + n)^y = z^2$. Burshtein[1] has considered an equation $7^x + 10^y = z^2$ of a similar form and proved that the equation has no solutions in positive integers x, y, z. Again Burshtein [9-10] in recent times, has also studied various diophantine equations of the similar form.

In this article, we study the Diophantine equations of the form

$$p^x + (p + n)^y = z^2 \text{ when } n = 3.$$

We consider two equations $3^x + 6^y = z^2$ and $5^x + 8^y = z^2$ with p as 3, 5 respectively.

2. Results

In this section, we find all the solutions to $3^x + 6^y = z^2$ and $5^x + 8^y = z^2$, in positive integers x, y, z.

Theorem 2.1. The equation

$$3^x + 6^y = z^2 \tag{1}$$

has finite solutions in positive integers x, y, z.

Proof: When $y = 1$, **we have the first solution as (1,1,3)** and this is the only solution possible as $3(3^{x-1} + 2)$ cannot be represented as a perfect square for any other value of x.
Considering $y > 1$,
As the left side of (1) is always odd, this indicates that z is odd.
As 3^x ends in 1,3,7 or 9, 6^y ends in 6, and z^2 ends in 1,5 or 9. For the left side of (1) to end in 1, 5 or 9, 3^x should end in 3 or 9 which indicates that x is either 1(mod 4) or 2(mod 4). Furthermore, considering mod 4 on (1), we have the right side to be 1(mod 4), and 6^y to be 0(mod 4) for any $y > 1$ and for 3^x to be 1(mod 4), x should be 2(mod 4).

Replacing x with $4m + 2$ in (1), we get

$$3^{4m+2} + 6^y = z^2 \tag{2}$$

Case (i): Considering y to be even, we have

$$3^{4m+2} + 6^{2n} = z^2$$

$$3^{4m+2} = z^2 - 6^{2n} = (z - 6^n)(z + 6^n) \tag{3}$$

Right side of (3) will be denoted as,

$$3^A = (z - 6^n), \qquad 3^B = (z + 6^n), \qquad A < B, \qquad A + B = 4m + 2.$$

We have, $3^B - 3^A = 2 \cdot 6^n$ which is $3^A(3^{B-A} - 1) = 2^{n+1}3^{2n}$.
As 3^A is odd and $3^{B-A} - 1$ is even,
we have,

$$3^{B-A} - 1 = 2^{n+1}3^{2n-A}$$

$$3^{B-A} - 2^{n+1}3^{2n-A} = 1 \tag{4}$$

As the left side of (4) is divisible by 3, so for the equation to hold, the power of 3 on the left side should be 1. Which makes (4) to be

$$3^{2n-A}(3^{B-2n} - 2^{n+1}) = 1, \tag{5}$$

In which, $3^{2n-A} = 1$.
So, $A = 2n$,(5) becomes $(3^{B-2n} - 2^{n+1}) = 1$.
The only positive solutions to the equation of the form,

$$3^x - 2^y = 1 \text{ are (1,1) and (2,3).} \tag{6}$$

Therefore, $B - 2n = 1, n + 1 = 1$ (or) $B - 2n = 2, n + 1 = 3$

Considering $B - 2n = 1, n + 1 = 1$, we obtain
$n = 0$, which indicates that $A = 0$, $B = 1$and $m = \frac{-1}{4}$. We don't have any solutions, as m is not an integer.

Considering $B - 2n = 2, n + 1 = 3$, we get $n = 2$, which indicates that $A = 4$, $B = 6$ and $m = 2$.
Here, we have the second solution as (6, 4, 45).

Case (ii): Considering y to be odd, we have

$$3^{4m+2} + 6^{2n+1} = z^2$$

$$6^{2n+1} = z^2 - 3^{4m+2} = (z + 3^{2m+1})(z - 3^{2m+1}) \tag{7}$$

Here we denote (7) as,

$z + 3^{2m+1} = 2^A 3^B, z - 3^{2m+1} = 2^C 3^D, \quad A + C = 2n + 1, B + D = 2n + 1.$

We have, $$2^A 3^B - 2^C 3^D = 2 \cdot 3^{2m+1}$$
$$2^{A-1} 3^{B-1} - 2^{C-1} 3^{D-1} = 3^{2m} \tag{8}$$

As 3^{2m} is odd, for the left side of (8) to be odd, one among 2^{A-1} and 2^{C-1} should be 1.

Consider $2^{A-1} = 1$, which implies that $A = 1$.

We rewrite (8) as, $$3^{B-1-2m} - 2^{C-1} 3^{D-1-2m} = 1$$
$$3^{D-1-2m}(3^{B-D} - 2^{C-1}) = 1$$

Here, $3^{D-1-2m} = 1$, $D = 1 + 2m$.

Then, we have $$3^{B-D} - 2^{C-1} = 1$$

According to (6), one set of solutions to the equation of this form is (1,1)
so, $B - D = 1$ and $C - 1 = 1$.
We have $C = 2$, $A = 2n - 1$, as $A = 1$ we have $n = 1$.
As $B + D = 2n + 1$ and $B - D = 1$, we get $B = 2$, $D = 1$ and $m = 0$.
Here, we have the third solution as (2,3,15).
The other set of solutions is (2,3).
so, $B - D = 2$ and $C - 1 = 3$.
We have $C = 4$, $A = 2n - 3$, as $A = 1$ we have $n = 2$. As $B + D = 2n + 1$ and $B - D = 2$, we get $B = \frac{7}{2}$, which is not an integer. Hence, there are no solutions.

Consider $2^{C-1} = 1$, which implies that $C = 1$.

We rewrite (8) as, $$2^{A-1} 3^{B-1-2m} - 3^{D-1-2m} = 1$$
$$3^{B-1-2m}(2^{A-1} - 3^{D-B}) = 1$$

Here, $3^{B-1-2m} = 1$, $B = 1 + 2m$.

Then we have, $$2^{A-1} - 3^{D-B} = 1 \tag{9}$$

Only solutions to the equation of the form (9) are $(1,0)$ and $(2,1)$.

Considering $A - 1$ =1 and $D - B$ =0, we have
$A = 1$, $B = D$, as $B + D = 2n + 1$, $B = \frac{2n+1}{2}$ which is not an integer, this implies that there are no solutions.

Considering $A - 1$ =2 and $D - B$ =1, we get
$A = 3$, $C = 1$, as $A + C = 2n + 1$, we have $A = 2n$, so $n = \frac{3}{2}$ which is not an integer. Hence, there are no solutions.

So, we have exactly 3 solutions to $3^x + 6^y = z^2$ in positive integers x, y, z, the solutions are (1,1,3), (2,3,15) and (6,4,45).

Theorem 2.2. The equation
$$5^x + 8^y = z^2 \tag{10}$$
has no solutions in positive integers x, y, z.

Proof: As the left side of (10) is always odd, this indicates that z is odd.
As 5^x ends in 5, 8^y ends in one among 2,4,6,8 and z^2 ends in 1,5,9. For the left side of (10) to end with 1, 5 or 9, 8^y should end in 4, 6. For 8^y to end in 4 or 6, y should be even.
As $z^2 \equiv 1(\text{mod } 8)$ for any odd value of z. For the left side of (10) to be 1(mod 8), 5^x should also be $\equiv 1(\text{mod } 8)$ as $8^y \equiv 0(\text{mod } 8)$. For 5^x to be $\equiv 1(\text{mod } 8)$, x should be even.

As both x, y are even, we represent $x = 2m$ and $y = 2n$. After replacing x with $2m$ and y with $2n$ in (10), we have

$$5^{2m} + 8^{2n} = z^2$$

which is, $$5^{2m} = z^2 - 8^{2n} = (z - 8^n)(z + 8^n) \qquad (11)$$

we denote the right side of (11) as,

$$z - 8^n = 5^A, \quad z + 8^n = 5^B, \quad A < B, \quad A + B = 2m.$$

Then $5^B - 5^A$ yields

$$5^A(5^{B-A} - 1) = 2 \cdot 8^n \qquad (12)$$

As $5^A(5^{B-A} - 1)$ is divisible by 5, whereas $2 \cdot 8^n$ is not, this indicates that $5^A = 1$. Therefore $A = 0$ in (12), and hence $B = 2m$. Then this implies that,

$$5^{2m} - 1 = 2 \cdot 8^n \qquad (13)$$

For all values of m, either $5^m - 1$ or $5^m + 1$ is divided by 3, whereas the right side of (13) cannot be represented as a multiple of 3.This implies that (13) cannot be true. Hence, we conclude that there are no positive values of x, y, z, that satisfies (10).

3. Conclusion

In this article, we proved that there are only finitely many solutions to the equation $3^x + 6^y = z^2$ and these solutions has one-one correspondence with the solutions of the equations$3^x - 2^y = 1$ and $2^x - 3^y = 1$. Furthermore, the equation $5^x + 8^y = z^2$ has no solutions in positive integers x, y, z.

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