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On Solutions to the Diophantine Equations $p^x + q^y = z^3$ when $p \geq 2, q$ are Primes and $1 \leq x,y \leq 2$ are Integers

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Abstract. In this article we consider the equations $p^x + q^y = z^3$ in which $p \geq 2$, q are distinct primes, z is a positive integer and the integers x, y satisfy $1 \leq x,y \leq 2$. The following three cases are examined, namely $x = y = 2$, $x = y = 1$, $x = 1$ and $y = 2$, when $p \geq 2$, q are primes. All the possibilities are determined for infinitely many solutions, unique solutions and cases of no solutions.

Keywords: Diophantine equations

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds.

In this article we consider the equations $p^x + q^y = z^3$ when $p \geq 2, q$ are primes and $1 \leq x,y \leq 2$ are integers. We investigate the three possibilities $x = y = 2$, $x = y = 1$, and $x = 1$, $y = 2$. For these possibilities we determine the cases of no solutions, infinitely many solutions, and unique solutions. This is done in the following Sections 2, 3 and 4, where each section and the theorems within are all self-contained.

2. All the solutions of $p^2 + q^2 = z^3$ when $p \geq 2, q$ are distinct primes

In this section, when p, q are distinct odd primes, it will be shown that $p^2 + q^2 = z^3$ has no solutions (Theorem 2.1). Whereas when $p = 2$, the equation $p^2 + q^2 = z^3$ has a unique solution (Theorem 2.2).

Theorem 2.1. Let z be a positive integer**.** If p, q are distinct odd primes, then the equation $p^2 + q^2 = z^3$ has no solutions.

Proof: We shall assume that $p^2 + q^2 = z^3$ has a solution, and reach a contradiction.

Since p, q are distinct odd primes, then by our assumption z^3 is even. Thus $z = 2T$ where T is an integer. Whether the prime p is of the form $4M + 3$ or of the form $4M + 1$ (M an integer), p^2 is of the form $4Q + 1$. Certainly, the same is true for any form of the prime q, and q^2 has the form $4R + 1$. Hence, $p^2 + q^2 = z^3$ yields

$$p^2 + q^2 = (4Q + 1) + (4R + 1) = 2(2Q + 2R + 1) \neq 8T^3 = z^3. \qquad (1)$$

It follows from (1) that our assumption is false, and the equation $p^2 + q^2 = z^3$ has no solutions when p, q are distinct odd primes.

The proof of Theorem 2.1 is complete. □

Theorem 2.2. Let z be a positive integer. If $p = 2$ and q is an odd prime, then the equation $2^2 + q^2 = z^3$ has the unique solution

$$2^2 + 11^2 = 5^3. \qquad (2)$$

Proof: We shall assume that $2^2 + q^2 = z^3$ has more than one solution and reach a contradiction.

The sum $2^2 + q^2$ is odd for all primes q. Then by our assumption z^3 is odd. Denote $z = 2T + 1$ where T is an integer. The values $T = 0, 1$ are clearly impossible, and hence $T \geq 2$. Whether $q = 4N + 3$ or $q = 4N + 1$ (N an integer), q^2 is of the form $4R + 1$. Hence z^3 is of the form $z^3 = 4 + q^2 = 4 + (4R + 1) = 4(R + 1) + 1$. If $T = 2U + 1$ (U an integer) is odd, then $z = 2T + 1 = 2(2U + 1) + 1 = 4U + 3$, and z^3 is of the form $4V + 3$ which is impossible. This implies that T is not odd, and therefore T is even. Denote $T = 2G$, where $G \geq 1$ is an integer and $z = 4G + 1$. Thus

$$z^3 = 64G^3 + 48G^2 + 12G + 1.$$

If $q = 4N + 1$, then

$$z^3 = 4 + (16N^2 + 8N + 1) = 4(4N^2 + 2N + 1) + 1 = 4G(16G^2 + 12G + 3) + 1$$

and after simplifications it follows that G is not even.
If $q = 4N + 3$, then

$$z^3 = 4 + (16N^2 + 24N + 9) = 4(4N^2 + 6N + 3) + 1 = 4G(16G^2 + 12G + 3) + 1$$

and after simplifications it follows that G is not even.
Hence G is odd. Denote $G = 2M + 1$, $M \geq 0$ is an integer, and $z = 8M + 5$.

When $M = 0$, the values $z = 5$ and $q = 11$ yield solution (2). We now show for all values $1 \leq M \leq 10$ or $13 \leq z \leq 85$ that the difference $z^3 - q^2 = 4$ is not achieved.

In the following Table 1, we consider for each value M the largest possible prime q in order to minimize the even difference $z^3 - q^2$.

Table 1:

M	$z = 8M + 5$	z^3	prime q	q^2	the difference $z^3 - q^2$
1	13	2197	43	1849	348
2	21	9261	89	7921	1340
3	29	24389	151	22801	1588
4	37	50653	223	49729	924
5	45	91125	293	85849	5276
6	53	148877	383	146689	2188
7	61	226981	467	218089	8892
8	69	328509	571	326041	2468
9	77	456533	673	452929	3604
10	85	614125	773	597529	16596

In Table 1, for all values $1 \leq M \leq 10$ the differences $z^3 - q^2$ are even, and each difference has at least 3 digits. If D denotes the number of digits of each such difference, then $D \geq 3$. Moreover, the value D is increasing, and hence $D = 1$ is never achieved. Therefore, for all values $M > 10$ with the respective values q, it is clear and self-evident that the difference $z^3 - q^2 = 4$ is never attained.

The uniqueness of solution (2) follows.

This concludes Theorem 2.2. □

3. All the solutions of the equation $2^1 + q^1 = z^3$ when q is prime

In this section we establish all the solutions of $2 + q = z^3$ when q is an odd prime. Moreover, we shall extend this equation to include odd composites. Hereafter, the odd value A will represent any odd prime or any odd composite.

Observe that all odd values A are either of the form $4N + 3$ or of the form $4N + 1$, and yield primes as well as composites. When $A = 4N + 3$, it is shown in Theorem 3.1 that $2 + A = z^3$ has infinitely many solutions. When $A = 4N + 1$, the same result is obtained in Theorem 3.2.

Theorem 3.1. Suppose that $N > 0$ and $M \geq 1$ are integers. Then, for each and every value M, there exists a unique value N satisfying the equation

$$2^1 + (4N + 3)^1 = (4M + 1)^3. \tag{3}$$

The equation $2^1 + (4N + 3)^1 = (4M + 1)^3$ has infinitely many solutions in which $4N + 3$ is prime, and when $4N + 3$ is composite.

Proof: Since $2 + (4N + 3) = 4(N + 1) + 1$, it is then justified to write that $z = 4M + 1$ in (3). From (3) after simplifications we obtain

$$N = 16M^3 + 12M^2 + 3M - 1. \tag{4}$$

It is clearly seen from (4) that every value N does not yield a value M. However, it follows from (4), that each and every value $M \geq 1$ determines a unique value N which satisfies (3).

The equation $2^1 + (4N + 3)^1 = (4M + 1)^3$ has infinitely many solutions.

This completes the proof of Theorem 3.1. □

Remark 3.1. The values $4N + 3$ represent composites and primes as well. When $M = 1, 2, 3,$ the first three solutions of (3) are exhibited in the following Table 2.

Table 2:

	Solutions of (3)	***N***	***M***	**4*N* + 3**
Solution 1	$2^1 + 123^1 = 5^3$	30	1	composite
Solution 2	$2^1 + 727^1 = 9^3$	181	2	Prime
Solution 3	$2^1 + 2195^1 = 13^3$	548	3	composite

Theorem 3.2. Suppose that $N > 0$ and $M \geq 0$ are integers. Then, for each and every value M, there exists a unique value N satisfying the equation

$$2^1 + (4N + 1)^1 = (4M + 3)^3. \quad (5)$$

The equation $2^1 + (4N + 1)^1 = (4M + 3)^3$ has infinitely many solutions in which $4N + 1$ is prime, and when $4N + 1$ is composite.

Proof: Since $2 + (4N + 1) = 4N + 3$, it is then justified to write that $z = 4M + 3$ in (5). From (5) after simplifications we have

$$N = 16M^3 + 36M^2 + 27M + 6. \quad (6)$$

Evidently, each value N does not yield a value M. But, (6) implies that each and every value $M \geq 0$ determines a unique value N which satisfies (5).

The equation $2^1 + (4N + 1)^1 = (4M + 3)^3$ has infinitely many solutions.

This concludes the proof of Theorem 3.2. □

Remark 3.2. The values $4N + 1$ represent composites as well as primes. When $M = 0, 1, 2, 3, 4,$ the first five solutions of (5) are demonstrated in Table 3.

Table 3:

	Solutions of (5)	***N***	***M***	**4*N* + 1**
Solution 4	$2^1 + 25^1 = 3^3$	6	0	composite
Solution 5	$2^1 + 341^1 = 7^3$	85	1	composite
Solution 6	$2^1 + 1329^1 = 11^3$	332	2	composite
Solution 7	$2^1 + 3373^1 = 15^3$	843	3	Prime
Solution 8	$2^1 + 6857^1 = 19^3$	1714	4	Prime

Concluding remark. Theorems 3.1 and 3.2, and the solutions demonstrated in Tables 2 and 3 imply that the equation $2^1 + A^1 = z^3$ with odd values A, has a unique solution for each and every odd value $z \geq 3$. The above equation has infinitely many solutions when A is prime and when A is composite.

4. All the solutions of the equation $2^1 + q^2 = z^3$ when q is prime

In this section we show (Theorem 4.1) when q is an odd prime that the equation $2 + q^2 = z^3$ has a unique solution.

Theorem 4.1. Let z be a positive integer. If q is an odd prime, then the equation $2 + q^2 = z^3$ has

(a) No solutions when q is of the form $4N + 3$.
(b) A unique solution when q is of the form $4N + 1$.

Proof: Observe that all odd primes q are either of the form $4N + 3$ or of the form $4N + 1$ where N is an integer. In both cases **(a)** and **(b)**, q^2 is of the form $4Q + 1$. Thus, $2 + q^2 = 4Q + 3$ implying that z^3 is odd. Since $(4M + 1)^3 \neq 4Q + 3$, it follows that $z = 4M + 3$ where $M \geq 0$ is an integer, and $z^3 = (4M + 3)^3$.

(a) Suppose that $q = 4N + 3$. Then we have

$$2 + (4N + 3)^2 = (4M + 3)^3. \quad (7)$$

In the following Table 4, we consider the first nine primes q where $3 \leq q \leq 59$.

Table 4:

N	$4N + 3 = q$	M	min $(4M + 3)$	The difference min $(4M + 3)^3 - (4N + 3)^2$
0	3	0	3	$3^3 - 3^2 = 18$
1	7	1	7	$7^3 - 7^2 = 294$
2	11	1	7	$7^3 - 11^2 = 222$
4	19	2	11	$11^3 - 19^2 = 970$
5	23	2	11	$11^3 - 23^2 = 802$
7	31	2	11	$11^3 - 31^2 = 370$
10	43	3	15	$15^3 - 43^2 = 1526$
11	47	3	15	$15^3 - 47^2 = 1166$
14	59	4	19	$19^3 - 59^2 = 3378$

To prove our assertion, it certainly suffices to use the first smallest possible value $(4M + 3)^3$ which exceeds $(4N + 3)^2$. We denote such a value by $\min(4M + 3)^3$. The following may now be observed from Table 4: (i) As required in (7), all differences are even, each of which is larger than the value 2. (ii) If D is the number of digits of each such difference, then as N increases, each value D is either equal or larger than

its predecessor value D. In any case $D \geq 2$. (iii) As a consequence of (i) and (ii), it follows that the value 2 in (7) is never attained.
Equation (7) has no solutions.

This completes part **(a)**.

(b) Suppose that $q = 4N + 1$. Then we have

$$2 + (4N + 1)^2 = (4M + 3)^3. \tag{8}$$

It has already been determined earlier that if $q = 4N + 1$, then q^2 is of the form $4Q + 1$. Thus, $2 + q^2 = 4Q + 3$ implying that z^3 is odd. Since $(4M + 1)^3 \neq 4Q + 3$, it follows that z must satisfy $z = 4M + 3$ where $M \geq 0$ is an integer, and $z^3 = (4M + 3)^3$ justifies (8).

An immediate and trivial solution of (8) is when $N = 1$ and $M = 0$, namely

$$2 + 5^2 = 3^3. \tag{9}$$

We will now show that solution (9) is unique.

In Table 5 we consider all primes $5 < q < 89$, where min $(4M + 3)^3$ is the first smallest possible value which exceeds $(4N + 1)^2$.

Table 5:

N	$4N + 1 = q$	M	min $(4M + 3)$	**The difference** min $(4M + 3)^3 - (4N + 1)^2$
3	13	1	7	$7^3 - 13^2 = 174$
4	17	1	7	$7^3 - 17^2 = 54$
7	29	2	11	$11^3 - 29^2 = 490$
9	37	3	15	$15^3 - 37^2 = 2006$
10	41	3	15	$15^3 - 41^2 = 1694$
13	53	3	15	$15^3 - 53^2 = 566$
15	61	4	19	$19^3 - 61^2 = 3138$
18	73	4	19	$19^3 - 73^2 = 1530$

It is self-evident from Table 5 that: (i) All differences are even as required, each of which is larger than the value 2. (ii) If D is the number of digits of each such difference, then $D \geq 2$. (iii) As a consequence of (i) and (ii), it follows that the value 2 in (8) is never attained.
Equation (8) has no solutions.

This concludes part **(b)** and Theorem 4.1. □

5. Conclusion

We have shown: (i) For distinct odd primes p, q the equation $p^2 + q^2 = z^3$ has no solutions, whereas when $p = 2$, the equation $2^2 + q^2 = z^3$ has a unique solution. (ii) When $2^1 + A^1 = z^3$ with odd values A, then for each and every odd value z there exists a value A prime or composite which satisfies the equation. (iii) When q is prime, then the equation $2^1 + q^2 = z^3$ has a unique solution.

Although Table 1 in Section 2, and Tables 4, 5 in Section 4 do not constitute a formal poof, nevertheless, the numbers presented in these tables speak for themselves, and strongly imply the validity of the statements of Theorem 2.2 and of Theorem 4.1.

We remark that to the best of our knowledge, other authors have not considered equations of the form $p^x + q^y = z^3$. It is therefore obvious, that there are no references on such an equation.

REFERENCES

1. N. Burshtein, All the solutions of the diophantine equation $p^x + p^y = z^4$ when $p \geq 2$ is prime and x, y, z are positive integers, *Annals of Pure and Applied Mathematics*, 21 (2) (2020) 125-128.
2. N. Burshtein, On the diophantine equations $p^4 + q^3 = z^2$ and $p^4 - q^3 = z^2$ when p, q are distinct odd primes, *Annals of Pure and Applied Mathematics*, 21 (1) (2020) 15-17.
3. N. Burshtein, The infinitude of solutions to the diophantine equation $p^3 + q = z^3$ when p, q are primes, *Annals of Pure and Applied Mathematics*, 17 (1) (2018) 135-136.
4. N. Burshtein, All the solutions of the diophantine equation $p^3 + q^2 = z^3$, *Annals of Pure and Applied Mathematics*, 14 (2) (2017) 207-211.
5. S. Chotchaisthit, On the diophantine equation $4^x + p^y = z^2$ where p is a prime number, *Amer. J. Math. Sci.*, 1 (1) (2012) 191-193.
6. S. Chowla, J. Cowles and M. Cowles, On $x^3 + y^3 = D$, *Journal of Number Theory*, 14 (1982) 369-373.
7. B. Poonen, Some diophantine equations of the form $x^n + y^n = z^m$, *Acta Arith.*, 86 (1998) 193-205.