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On Structure of Order Graphs Arising from Groups

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Abstract. Graph theory is a helpful tool in studying group properties. There has been a strong relationship between group (finite group) and graph theories for more than a century. The order graph of a finite group is the graph (undirected) whose vertices are the elements of the group and for any two distinct vertices H and K there is an edge from H to K, if and only if, order of H divides order of K (or vice-versa). In this paper, we focus on the interplay between the group-theoretic properties of G (finite group) and the graph-theoretic properties of $\Gamma(G)$ (undirected graph).

Keywords: Finite group, order graph, star graph

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1. Introduction

Graph theory is the study of graphs, which are mathematical structures, used to model pair wise relations between objects. Group is a set of objects with a rule of combination. Given any two elements of the group, the rule yields another group element depending upon the elements chosen. The information in a group can be represented by a graph, which is a collection of points, called “vertices” and lines between them, called “edges”. There are many ways to establish a link between graph and a finite group, which results into many of the group properties. The association between a graph and a group (finite) is usually determined by the adjacency of the vertices. To be more specific, to make the edges we pick some elements from the group. The combinatorial properties of graphs have been employed to investigate the theoretic algebraic properties of groups and vice versa. The relationship between a graph and a group (finite) was first introduced by Arthur Cayley in 1878 [1,7] in which a graph represents a finite group. Cayley graphs geometrically display the actions of a finite group. There are some other well-known graphs associated with finite groups such as the power graph [6]. It is the graph of a group G which has all elements of G as vertices, such that two distinct vertices are adjacent if and only if, one of them is a power of the other. Prime graphs, came into existence as a by-product of some cohomological questions raised by K.W. Gruenberg in the 1970s [10]. The order graph of a finite group is the undirected graph whose vertices are the elements of the group (finite) and for any two distinct vertices there is an edge if and only if, one divides the other. The main objective of the paper is to focus on the structure of order graphs with its basic results. The paper is organized as follows. In Section 2, we recall some basic definitions of graphs. Section 3 and Section 4 respectively deals with order graphs, Sub-graph of a order graph with their basic properties.

2. Pre-requisites

Throughout this paper, G denotes a finite group and Γ represents an undirected graph. Let $o(G)$ denotes the order of G, the number of elements in a group. A group G is said to be a p-group, p-prime, if order of every element of the group G is the power of prime p.

Definition 2.1. A graph Γ is a triple (V, E, φ) consisting of a non-empty set $V(\Gamma)$ of vertices, a set $E(\Gamma)$ of edges and an incidence function φ that associates with each edge of Γ a pair of vertices (not necessarily distinct) of Γ.

Consider an undirected graph $\Gamma(V, E, \varphi)$. Then we have the following observations.

- Let $x, y \in V(\Gamma)$ are said to be adjacent, if there is an edge e, so that $\varphi(e) = (x, y)$ or $\varphi(e) = (y, x)$. An edge $e \in E$ is called a loop (self loop), if $\varphi(e) = (x, x)$ for $x \in V(\Gamma)$

- A sub graph $\Omega(V^*, E^*, \varphi^*)$ of a graph $\Gamma(V, E, \varphi)$ is a graph in which $V^* \subseteq V$, $E^* \subseteq E$.

- The graph Γ has multiple edges, if there exist two edges e and f in $E(\Gamma)$, such that $\varphi(e) = \varphi(f) = (x, y)$ for $x, y \in V(\Gamma)$.If $x \in V(\Gamma)$, then the degree of x, denoted by $\deg(x)$, is the number of edges incident with it.

- The graph Γ is a simple graph, if it has no loops or multiple edges. A path between two elements $x, y \in V(\Gamma)$ is a sequence of edges which connect a sequence of distinct vertices. The number of edges in a path is called the length of a path.

- The distance between $x, y \in V(\Gamma)$ denoted by $d(x, y)$, is the length of a shortest path between x and y. The diameter of a graph Γ is denoted by $diam(\Gamma)$ and is defined to be $diam(\Gamma) = \max\{d(x, y) : x, y \in V(\Gamma)\}$.

- A cycle is a non-trivial path in a graph from a vertex to itself. The girth of graph Γ, denoted by $gr(\Gamma)$ and is defined to be the length of shortest cycle in Γ, $gr(\Gamma) = \infty$ if Γ contains no cycles.

- A graph Γ is said to be connected, if there is a path between any two distinct vertices, otherwise it is disconnected graph.

- A complete graph is a simple graph whose vertices are pair-wise adjacent; the complete graph with n vertices is denoted by K_n.

- A graph Γ is called complete bipartite, if there exist disjoint subsets A, B of $V(\Gamma)$ such that $A \cup B = V(\Gamma)$ and (x, y) for any $x \in A$ and $y \in B$. When graph Γ is finite, complete bipartite graphs are denoted by $K_{m,n}$; where $|A| = m$ and $|B| = n$. A star graph if given by $\Gamma = K_{1,n}$

3. Order graphs and their properties

Definition 3.1. Let G be a group. The order graph of G is the graph (undirected) $\Gamma(G)$, whose vertices are non-trivial sub-groups of G and for two distinct vertices H and K there is an edge from H to K, if and only if either $o(H)/o(K)$ or $o(K)/o(H)$

The order graphs of Z_8 and Z_{10} are as follows.

Example 3.1. Let $G = Z_8$ be the cyclic group of order 8 then, vertex set $V = \{1,2,4,8\}$ and edge set $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ where $\varphi(e_1) = (1,2)$, $\varphi(e_2) = (1,4)$, $\varphi(e_3) = (1,8)$, $\varphi(e_4) = (2,8)$, $\varphi(e_5) = (4,8)$ and $\varphi(e_6) = (2,4)$

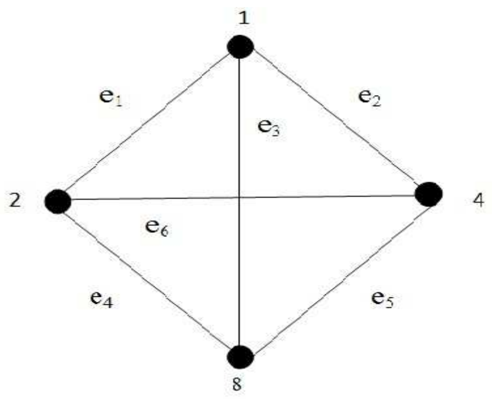

Figure 1:

Example 3.2. Let $G = Z_{10}$ be a cyclic group of order 10. Then vertex set $V = \{1,2,5,10\}$ and edge set $E = \{e_1, e_2, e_3, e_4, e_5\}$ where $\varphi(e_1) = (1,2)$, $\varphi(e_2) = (1,5)$, $\varphi(e_3) = (1,10)$, $\varphi(e_4) = (2,10)$ and $\varphi(e_5) = (5,10)$

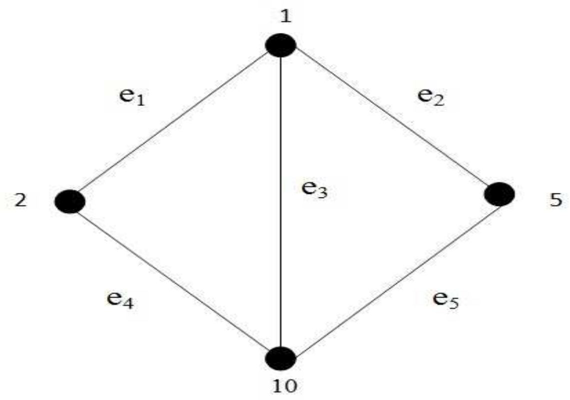

Figure 2:

Lemma 3.1. For any finite group G, there exist a vertex of $\Gamma(G)$ which is adjacent to every other vertices.

Proof: Let H be a subgroup of G, then by Lagrange's Theorem [4] [7], $o(H)/o(G)$ (order of a subgroup divides order of a group), So G has a vertex of $\Gamma(G)$, which is adjacent to every other vertices.

Corollary 3.1. Let G be a finite group, then $\Gamma(G)=\phi$, if and only if G is a simple group. Further, $\Gamma(G)\neq\phi$; if and only if $\Gamma(G)$ has not a single vertex.

Corollary 3.2. Let G be a finite group. Then $\Gamma(G)$ is a regular graph, if and only if, it is complete.

Proof: Let $\Gamma(G)$ be a graph with n-vertices then maximum degree of $\Gamma(G)$, $\Delta(\Gamma(G))=n-1$

Hence, $\Gamma(G)$ is regular, if the degree of any other vertices is equal to $n-1$, which implies that $\Gamma(G)$ is complete.

Lemma 3.2. Let G be a finite group. Then $\Gamma(G)$ is connected graph with $diam\Gamma(G)\leq 2$

Proof: By Lemma **3.1.**, $\Gamma(G)$ has a vertex adjacent to every other vertices. So that, $\Gamma(G)$ is connected graph and hence $diam\Gamma(G)\leq 2$

Theorem 3.1. Let G be a finite group. Then $\Gamma(G)$ is complete graph, if and only if, G is a p-group, where p is a prime number.

Proof: Let $\Gamma(G)$ be a complete graph and $o(G)=pqn$, where p,q are prime numbers and n is a natural number. By Cauchy's Theorem, there are subgroups H and K of G

with $o(H)=p$ and $o(K)=q$. By assumption we must have p/q, it follows that $p=q$. Thus, G is a p-group.
Conversely, let G be a p-group. Then the orders of all subgroups of G is a prime power for some p (p-prime number) so that, all vertices of $\Gamma(G)$ are adjacent with each others. Thus, $\Gamma(G)$ is a complete graph.
Let G be a finite group. The order graph $\Gamma(G)$ is a complete bipartite graph, whenever $\Gamma(G)$ is a star graph whose vertex is adjacent with other vertices.

Lemma 3.3. Let G be a finite group. Then $\Gamma(G)=K_{1,n}$ $(n\geq 2)$, if and only if any subgroups of G has prime order and G has only one subgroup with any orders.

Lemma 3.4. Let G be a finite group. Then $\Gamma(G)$ is a star graph, if and only if $G=C_{pq}$ where p, q are prime numbers.

4. A subgraph of a order graph

Let G be a finite group. The sub-graph of $\Gamma(G)$ denoted by $\Omega(G)$ with vertex set given by $V(\Omega(G))=V(\Gamma(G))-\{G\}$; so that $\Omega(G)$ is a simple graph with vertices $V(\Omega(G))=\{H\}$, where H is a non-trivial proper subgroup of G and two distinct vertices H and K are adjacent, if and only if either $o(H)/o(K)$ or $o(K)/o(H)$.

Theorem 4.1. Let G be a finite group. Then $\Omega(G)$ is a connected graph with diameter at most four.
Proof: Let H and K be two proper subgroups of G, then there must be a path from H to K in $\Omega(G)$.If $o(H).o(K)>o(G)$ then $o(H\cap K)=m$, for $m>1$ there is path from $H-H\cap K-K$
If $o(G)>o(H).o(K)$ then HK is a proper subgroup of G. Then there will be path $H-HK-K$
Now, let $o(G)=o(H).o(K)$ and If $o(H\cap K)=m$, for $m>1$ then we are done.
Suppose $o(H\cap K)=1$, then there are prime numbers p,q such that $p/o(H)$ and $q/o(K)$
If $o(G)=pq$, then $\Omega(G)=\phi$ then we are done. Assume that , $o(G)>pq$ then clearly G has three subgroups which establishes a path between $H-K$. So $\Omega(G)$ is a connected graph and diameter at most four, $diam(\Omega)\leq 4$.

Corollary 4.1. Let G be a finite group and H be a subgroup of G with $o(H)=p^n m$, where p is a prime number, $n\geq 2$ and $m>1$ then $gr(\Omega(G))=3$.

Theorem 4.2. Let G be a finite abelian group such that $\Omega(G) \neq \phi$; ($\Omega(G)$; being the sub-graph of a order graph) Then $\Omega(G)$ is a complete graph if and only if, G is a p-group.

Proof: Suppose that $o(G) = pqn$, for some prime numbers p, q & $n \geq 1$. Then G has at least three proper subgroups with orders $o(H) = p$, $o(K) = q$ and $o(L) = pq$, it follows that H, K are not adjacent to each other which is contradiction to the supposition $\Omega(G)$ is a complete graph. So, either $m = 1$ or $p = q$. If $m = 1$ and $p \neq q$ then $G \cong C_{pq}$ and $\Omega(G) = \phi$, which is a contradiction to assumption. Hence $p = q$. By continuing this way, one can show that G is a p-group.

Lemma 4.1. Let G be a finite group with $o(G) = pq$, $(p < q)$ where p, q are distinct prime numbers, then either $\Omega(G) = \phi$ or $\Omega(G) = K_n$, complete graph with n-vertices.

Lemma 4.2. Let G be a finite group. Then, $\Omega(G)$ is not a star graph.
For a finite group G, one can show that $\Omega(G) \neq K_{1,n}$ for $n \geq 2$.

Conclusion 5. The paper is an attempt to highlight the association of graphs and finite group, called order graphs and sub-graph of a order graph. One can extend the concept to investigate some underlying structures of order graphs for further study.

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REFERENCES

1. N.Biggs, *Algebraic Graph Theory,* Cambridge University Press, Cambridge (1993).
2. N.Bilal, Al-Hasanat et.al., Groups with prime order classes, *International Journal of Pure and Applied Mathematics*, 116 (2017) 695-699.
3. N.Bilal, Al-Hasanat and Ahmad S. Al-Hasant, Order graph: A new representation of finite groups, *International Journal of Mathematics and Computer Science,* 14 (2019) 809-819.
4. H.E.Rose, *A Course on Finite Groups;* Cambridge University Press Cambridge (1978).
5. I.M.Isaacs, *Finite Group Theory, Graduate Studies in Mathematics,* American Mathematical Society, Providence (2008).
6. J.Rose, *A Course on Group Theory*, Cambridge University Press, Cambridge (1998).
7. S.H.Payrovi and H.Pasebani, The order Graphs of Groups, *Journal of Algebraic Structures and Their Applications,* (2014) 1-10.
8. V.Kelarev and S.J.Quinn, A combinatorial property and power graphs of groups, *Contributions to General Algebra*, 12 (2000) 3-6.
9. D.S.Dummit and R.M.Foote, *Abstract Algebra*, Third edition, John Wiley and Sons, Inc., Hoboken, NJ, (2004).

10. E.A.Bertram, Some applications of graph theory to finite groups, *Discrete Math.*, 44 (1983) 31–43.
11. M.Fang and P.Zhang, Finite groups with graphs containing no triangles, *J. Algebra*, 264 (2003) 613–619.
12. E.Konstantinova, Vertex reconstruction in Cayley graphs, *Discrete Math.*, 309 (2009) 548–559
13. M.Mirzargar, P.P.Pach and A.R.Ashrafi, Remarks on commuting graph of a finite group, *Electron. Notes Discrete Math.*, 45 (2014) 103–106.
14. M.Pal, Intersection graphs: an introduction, *Annals of Pure and Applied Mathematics*, 4(1) (2013) 43-91.
15. V.S.Shigehalli and G.V.Uppin, Some properties of glue graph, *Annals of Pure and Applied Mathematics*, 6(1) (2014) 98-103.
16. B.Jana and S.Mondal, Computation of a minimum average distance tree on permutation graphs, *Annals of Pure and Applied Mathematics*, 2(1) (2012) 74-85.