Annals of Pure and Applied Mathematics
Vol. 22, No. 1, 2020, 29-39
ISSN: 2279-087X (P), 2279-0888(online)
Published on 17 August 2020
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v22n1a05678

Annals of
Pure and Applied Mathematics

Numerical Approximation of Surface Integrals Using Mixed Cubature Adaptive Scheme

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Received 4 July 2020; accepted 2 August 2020

Abstract. This research described the development of a new mixed cubature rule for evaluation of surface integrals over rectangular domains. Taking the linear combination of Clenshaw-Curtis 5- point rule and Gauss-Legendre 3-point rule (each rule is of same precision i.e. precision 5) in two dimensions the mixed cubature rule of higher precision was formed (i.e. precision 7). This method is iterative in nature and relies on the function values at uneven spaced points on the rectangle of integration. Also as supplement, an adaptive cubature algorithm is designed in order to reinforce our mixed cubature rule. With the illustration of numerical examples this mixed cubature rule is turned out to be more powerful when compared with the constituents standard cubature procedures both in adaptive and non-adaptive environment.

Keywords: Clenshaw-Curtis 5-point cubature rule, Gauss-Legendre 3-point rule, mixed cubature rule, adaptive cubature scheme

AMS Mathematics Subject Classification (MSC2020): 65D30, 65D32

1. Introduction

Of the front, in numerical analysis, after classical quadrature method (Newton Cotes), for evaluation of real definite integrals, Gaussian Quadrature [1,2,3,4,5] method and Clenshaw-Curtis quadrature method [2,12] are remarkable benchmarks. Clenshaw-Curtis quadrature integrates the function over the zeros of the Chebyshev polynomial where as the Gaussian quadrature integrates a function over the zeros of several orthogonal polynomials (Gauss-Legendre over the roots of the Legendre polynomial, Gauss Hermite over the roots of the Hermite polynomial etc.). As we know an n-point Gaussian rule is of precision $2n-1$, where as the precision of an n-point Clenshaw-Curtis rule is n. In general, Gauss type rule is of higher precision than that of Clenshaw-Curtis type when same abscissa are used.

Two names came to the very front, Das and Pradhan [6], the men who came forth and brilliantly brought out a new quadrature method known as "Mixed Quadrature" in 1996. The mixed quadrature rule involves construction of symmetric quadrature rule of higher precision as a linear/ convex combination of two other rules of equal lower precision. At first they formed this rule by combining Simpson's $\frac{1}{3}$ rule and Gauss-Legendre 2-point rule in one variable, where each rule is of precision 3. The new quadrature rule not only found to be effective with higher precision (i.e., precision 5) but also showed its superiority on the constituent rules with the evaluation of some real definite integrals numerically.

Down the years many authors [7-10] were marvelous in evaluating real definite integrals and also integrals of analytic functions in one dimension applying mixed quadrature rule. But the process did not stop here. Again there was a question, what happens, if an integrand takes a sharp pick within the nodes of the quadrature rule? Can we still get a better approximation over the prescribed interval using mixed quadrature process?

Surprisingly a much better answer with a much better quadrature process eventuated by Dash and Das. For the first time they capitalized an adaptive integration process [12,13,14] by fixing up a termination criterion to strengthen up the mixed quadrature rule for approximate evaluation of real definite integrals in one dimension.

With functions of two or more variables not only the function can cause the integral to be difficult, but also the region over which we integrate. A region of $\mathbb{R}^2$ can have any shape. Even if the function is easy to integrate, if the region is complex enough, the integral will still be difficult to evaluate. It is then flexible enough to take into account only rectangular regions. Integrating the functions over rectangle, the error incurred by making approximation to the integrals can be bounded or estimated approximately.

Let us assume we are given a function *f(x, y)* defined on a closed and bounded region **D.** Because **D** is bounded, it can be enclosed in a rectangle **R**. i.e.,

Integrals of the type

$$I(f)=\iint_R f(x,y)\,dxdy \tag{1.1}$$

can be evaluated with in a closed rectangle $[a,b]\times[c,d]$ i.e.,

$$I(f)=\iint_R f(x,y)\,dxdy=\int_a^b\int_c^d f(x,y)\,dxdy \tag{1.2}$$

This closed rectangle $[a,b]\times[c,d]$ can be transformed into a standard 2 square $[-1,1]\times[-1,1]$ as an integrating domain i.e.

$$\int_a^b\int_c^d f(x,y)\,dxdy=\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dxdy \tag{1.3}$$

The integrals of the type (1.3) have been successfully approximated by some authors [11,15] using mixed quadrature rule.

Once again a question strikes the mind, is it possible to find an integration scheme which can treat the ill behavior of the integrand in (1.3) delicately over the domain [*a*, *b*] × [*c*, *d*]. Means if *f* (*x*, *y*), besides continuous, is irregular or badly shaped then can we make a smooth or productive approach in order to accomplish a much better approximation to the integrand? Once again the answer is favorable.

In this paper, Patra, Das and Dash are first to revitalize the mixed quadrature process with the outfit of adaptive quadrature algorithm in two dimensions. Keeping in view the improvement of precision. At first we have formulate a mixed cubature rule of precision seven by blending Clenshaw-Curtis 5- point rule and Gauss-Legendre 3-point rule in two dimensions each is of precision five. The theoretical dominance of this cubature rule over its constituent rules is established through error analysis. Then we have designed an adaptive quadrature algorithm taking the mixed cubature rule $R^2_{CC_5GL_3}(f)$ as the base rule. This adaptive integration scheme not only outstanding in giving better approximation with a specified tolerance but also it helps to evaluate $f(x, y)$ very few times. Literally, we can say the number of steps the integrand requires to reach the accuracy is lessen up.

One can see

(i) The analytical comparison of the mixed cubature rule with their constituent rules (using non adaptive scheme) given in table-1.

(ii) The analytical comparison of the mixed cubature rule with the mixed cubature of Simpson's $\frac{1}{3}$rd rule and Gauss-Legendre 2-point rule $\left(R^2_{S_3GL_2}(f)\right)$ developed in a previous paper[15] (using non adaptive scheme) given in table-2.

(iii) The analytical comparison of the mixed cubature rule with their constituent rules (using adaptive scheme) given in table-3

(iv) The analytical comparison of the mixed cubature rule with the mixed cubature rule $\left(R^2_{S_3GL_2}(f)\right)$ (using adaptive scheme) given in table-4.

2. Formulation of mixed cubature rule of precision seven in two dimensions

The Clenshaw-Curtis 5-point rule in one dimension is

$$I(f)=\int_{-1}^{1} f(x)\,dx \approx R_{CC_5}(f)=\frac{1}{15}\left[f(-1)+8f\left(-\frac{1}{\sqrt{2}}\right)+12f(0)+8f\left(\frac{1}{\sqrt{2}}\right)+f(1)\right] \tag{2.1}$$

So we can reformulate Clenshaw-Curtis 5-point rule in two dimensions as

$$\begin{aligned}
I(f) &= \int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dxdy \approx R^2_{CC_5}(f)\\
&= \frac{1}{15^2}\left[f(-1,-1)+8f\left(-1,-\frac{1}{\sqrt{2}}\right)+12f(-1,0)+8f\left(-1,\frac{1}{\sqrt{2}}\right)+f(-1,1)\right]\\
&\quad+\frac{8}{15^2}\left[f\left(-\frac{1}{\sqrt{2}},-1\right)+8f\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)+12f\left(-\frac{1}{\sqrt{2}},0\right)+8f\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)+f\left(-\frac{1}{\sqrt{2}},1\right)\right]\\
&\quad+\frac{12}{15^2}\left[f(0,-1)+8f\left(0,-\frac{1}{\sqrt{2}}\right)+12f(0,0)+8f\left(0,\frac{1}{\sqrt{2}}\right)+f(0,1)\right]\\
&\quad+\frac{8}{15^2}\left[f\left(\frac{1}{\sqrt{2}},-1\right)+8f\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)+12f\left(\frac{1}{\sqrt{2}},0\right)+8f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)+f\left(\frac{1}{\sqrt{2}},1\right)\right]\\
&\quad+\frac{1}{15^2}\left[f(1,-1)+8f\left(1,-\frac{1}{\sqrt{2}}\right)+12f(1,0)+8f\left(1,\frac{1}{\sqrt{2}}\right)+f(1,1)\right]
\end{aligned} \tag{2.2}$$

and the Gauss-Legendre 3-point rule in one dimension is

$$I(f)=\int_{-1}^{1} f(x)\,dx \approx R_{GL_3}(f)=\frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right)+8f(0)+5f\left(\sqrt{\frac{3}{5}}\right)\right] \tag{2.3}$$

the reformulation of Gauss-Legendre 3-point rule in two dimensions gives

$$\begin{aligned} I(f)&=\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dxdy \approx R^2_{GL_3}(f)\\ &=\frac{5}{9^2}\left[5f\left(-\sqrt{\frac{3}{5}},-\sqrt{\frac{3}{5}}\right)+8f\left(-\sqrt{\frac{3}{5}},0\right)+5f\left(-\sqrt{\frac{3}{5}},\sqrt{\frac{3}{5}}\right)\right]\\ &+\frac{8}{9^2}\left[5f\left(0,-\sqrt{\frac{3}{5}}\right)+8f(0,0)+5f\left(0,\sqrt{\frac{3}{5}}\right)\right]\\ &+\frac{5}{9^2}\left[5f\left(\sqrt{\frac{3}{5}},-\sqrt{\frac{3}{5}}\right)+8f\left(\sqrt{\frac{3}{5}},0\right)+5f\left(\sqrt{\frac{3}{5}},\sqrt{\frac{3}{5}}\right)\right] \end{aligned} \tag{2.4}$$

Let $E^2_{CC_5}(f)$ and $E^2_{GL_3}(f)$ denote the error terms in approximating the integral $I(f)$ by the rules (2.2) and (2.4) respectively. Let

$$I(f)=R^2_{CC_5}(f)+E^2_{CC_5}(f) \tag{2.5}$$

$$I(f)=R^2_{GL_3}(f)+E^2_{GL_3}(f) \tag{2.6}$$

Using Maclaurin's expansion of functions in two variables we get from equations (2.5) and (2.6) we get

$$\begin{aligned} E^2_{CC_5}(f)&=\frac{1}{18900}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+\frac{1}{907200}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]\\ &+\frac{1}{113400}\left[f_{6,2}(0,0)+f_{2,6}(0,0)\right]+\cdots \end{aligned} \tag{2.7}$$

$$\begin{aligned} E^2_{GL_3}(f)&=\frac{1}{7875}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+\frac{11}{2835000}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]\\ &+\frac{1}{47250}\left[f_{6,2}(0,0)+f_{2,6}(0,0)\right]+\cdots \end{aligned} \tag{2.8}$$

This shows that the rules (2.2) and (2.4) are of precision 5.

Now multiplying the equations (2.5) and (2.6) by $\frac{1}{5}$ and $-\frac{1}{12}$ respectively, and then adding the resulting equations we obtain

$$I(f)=\frac{1}{7}\left[12R^2_{CC_5}(f)-5R^2_{GL_3}(f)\right]+\frac{1}{7}\left[12E^2_{CC_5}(f)-5E^2_{GL_3}(f)\right]$$

or $$I(f)=R^2_{CC_5GL_3}(f)+E^2_{CC_5GL_3}(f) \tag{2.9}$$

where $$R^2_{CC_5GL_3}(f)=\frac{1}{7}\left[12R^2_{CC_5}(f)-5R^2_{GL_3}(f)\right] \tag{2.10}$$

This is the desired mixed cubature rule of precision seven for approximate evaluation of $I(f)$. The truncation error generated in this approximation is given by

$$E^2_{CC_5GL_3}(f)=\frac{1}{7}\left[12E^2_{CC_5}(f)-5E^2_{GL_3}(f)\right]$$

$$= -\frac{1}{1134000}\left[f_{8,0}(0,0) + f_{0,8}(0,0) \right] + \cdots \tag{2.11}$$

The rule (2.10) may be called as a mixed type rule as it is constructed from two different types of rules of the same precision (i.e. precision 5)

3. Error analysis

An asymptotic error estimate and error bound of the rule (2.10) are given in theorems (3.1) and (3.2) respectively.

Theorem 3.1. Let $f(x,y)$ be a continuously differentiable function in the closed rectangle $[-1,1]\times[-1,1]$. Then the error $E^2_{CC_5GL_3}(f)$ associated with the rule $R^2_{CC_5GL_3}(f)$ is given by

$$\left|E^2_{CC_5GL_3}(f)\right| \approx \frac{1}{1134000}\left|\left[f_{8,0}(0,0) + f_{0,8}(0,0) \right]\right|$$

Proof: Follows immediately from equation (2.11).

Theorem 3.2. The bound of the truncation error $E^2_{CC_5GL_3}(f) = I(f) - R^2_{CC_5GL_3}(f)$ is given by

$$\left|E^2_{CC_5GL_3}(f)\right| \le \frac{M}{11025}|\xi_2 - \xi_1| \times |\eta_2 - \eta_1|$$

$$M = \max_{\substack{-1\le x\le 1\\ -1\le y\le 1}} \left|\left[f_{7,0}(x,0) + f_{0,7}(0,y) \right]\right|$$

Proof: We have from (2.7) and (2.8)

$$E^2_{CC_5}(f) \approx \frac{1}{18900}\left[f_{6,0}(\xi_2,\eta_2) + f_{0,6}(\xi_2,\eta_2) \right], (\xi_2,\eta_2) \in [-1,1]\times[-1,1]$$

$$E^2_{GL_3}(f) \approx \frac{1}{7875}\left[f_{6,0}(\xi_1,\eta_1) + f_{0,6}(\xi_1,\eta_1) \right], (\xi_1,\eta_1) \in [-1,1]\times[-1,1]$$

We know

$$\begin{aligned} E^2_{CC_5GL_3}(f) &= \frac{1}{7}\left[12E^2_{CC_5}(f) - 5E^2_{GL_3}(f) \right] \\ &\approx \frac{1}{7}\left[\frac{1}{1575}\left\{ f_{6,0}(\xi_2,0) + f_{0,6}(0,\eta_2) \right\} - \frac{1}{1575}\left\{ f_{6,0}(\xi_1,0) + f_{0,6}(0,\eta_1) \right\} \right] \\ &= \frac{1}{11025}\left[\left\{ f_{6,0}(\xi_2,0) + f_{0,6}(0,\eta_2) \right\} - \left\{ f_{6,0}(\xi_1,0) + f_{0,6}(0,\eta_1) \right\} \right] \end{aligned}$$

$$= \frac{1}{11025}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}\left[f_{7,0}(x,0) + f_{0,7}(0,y) \right] dxdy \quad (\text{assuming } \xi_1 < \xi_2 \text{ and } \eta_1 < \eta_2)$$

so $$\left|E^2_{CC_5GL}(f)\right| \approx \left|\frac{1}{11025}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}\left[f_{7,0}(x,0) + f_{0,7}(0,y) \right] dxdy\right|$$

$$\le \frac{1}{11025}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}\left|\left[f_{7,0}(x,0)+f_{0,7}(0,y)\right]\right|dxdy$$

and so $\left|E^2_{CC_5GL_3}(f)\right| \le \frac{1}{11025} M \int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2} dxdy$

where $M = \max\limits_{\substack{-1\le x\le 1\\ -1\le y\le 1}}\left|\left[f_{7,0}(x,0)+f_{0,7}(0,y)\right]\right|$

$$= \frac{M}{11025}\left|(\xi_2-\xi_1)\times(\eta_2-\eta_1)\right|$$

which gives only a theoretical error bound as $(\xi_1-\eta_1)$ and $(\xi_2-\eta_2)$ are unknown points in $[-1,1]\times[-1,1]$. It shows that the error in the approximation will be less if the points $(\xi_2,\eta_2),(\xi_1,\eta_1)$ get closed to each other.

Corollary 3.1. The error bound for the truncation error $E^2_{CC_5GL_3}(f)$ is given by

$$\left|E^2_{CC_5GL_3}(f)\right| \le \frac{4M}{11025}$$

Proof: We know from theorem (3.2) that

$$\left|E^2_{CC_5GL_3}(f)\right| \le \frac{M}{11025}\left|(\xi_2-\xi_1)\times(\eta_2-\eta_1)\right|,\ (\xi_2,\eta_2),(\xi_1,\eta_1)\in[-1,1]\times[-1,1]$$

where $M = \max\limits_{\substack{-1\le x\le 1\\ -1\le y\le 1}}\left|\left[f_{7,0}(x,0)+f_{0,7}(0,y)\right]\right|$

choosing $\left|(\xi_2-\xi_1)\right|\le 2$ and $\left|(\eta_2-\eta_1)\right|\le 2$

we get $\left|E^2_{CC_5GL_3}(f)\right| \le \frac{4M}{11025}$

4. Adaptive cubature algorithm for evaluation of surface integrals

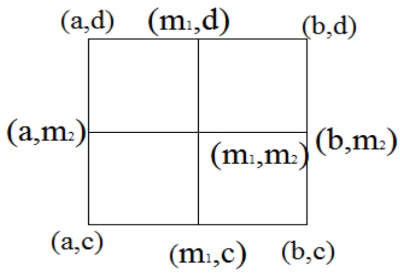
To evaluate surface integrals over any rectangle $\{[a,b]\times[c,d]\}$ using adaptive cubature scheme, we adopt the following four steps algorithm.

Input: Function $f:[a,b]\times[c,d]\to\mathbb{R}$ and the prescribed tolerance ε.

Output: An approximation $Q(f)$ to the integral $I(f)=\int_a^b\int_c^d f(x,y)\,dxdy$ such that $\left|Q(f)-I(f)\right|\le\varepsilon$

Step-1: The mixed cubature rule $R^2_{CC_5GL_3}(f)$ is applied over the rectangle $[a,b]\times[c,d]$ having corner points $(a,c),(b,c),(b,d)$ and (a,d) to approximate the surface integral $I(f)=\int_a^b\int_c^d f(x,y)\,dxdy$. The approximated value is denoted by $R^2_{CC_5GL_3}\left(f_{[a,b]\times[c,d]}\right)$.

Step-2: The rectangle of integration $[a,b]\times[c,d]$ is split into four equal pieces of rectangles A_1, A_2, A_3 and A_4 having corner points $\{(a,c),(m_1,c),(m_1,m_2),(a,m_2)\}$, $\{(m_1,c),(b,c),(b,m_2),(m_1,m_2)\}$, $\{(m_1,m_2)(b,m_2),(b,d),(m_1,d)\}$ and $\{(a,m_2),(m_1,m_2)(m_1,d),(a,d)\}$ respectively, where $m_1=\frac{a+b}{2}$ and $m_2=\frac{c+d}{2}$.



The mixed cubature rule $R^2_{CC_5GL_3}(f)$ is applied over each small rectangle to approximate the surface integrals $I_1(f)=\int_a^{m_1}\int_c^{m_2} f(x,y)\,dxdy$, $I_2(f)=\int_{m_1}^{b}\int_c^{m_2} f(x,y)\,dxdy$, $I_3(f)=\int_{m_1}^{b}\int_{m_2}^{d} f(x,y)\,dxdy$ and $I_4(f)=\int_a^{m_1}\int_{m_2}^{d} f(x,y)\,dxdy$ respectively. The approximated values are denoted by $R^2_{CC_5GL_3}\left(f_{[a,m_1]\times[c,m_2]}\right)$, $R^2_{CC_5GL_3}\left(f_{[m_1,b]\times[c,m_2]}\right)$, $R^2_{CC_5GL_3}\left(f_{[m_1,b]\times[m_2,d]}\right)$ and $R^2_{CC_5GL_3}\left(f_{[a,m_1]\times[m_2,d]}\right)$ respectively.

Step-3: $R^2_{CC_5GL_3}\left(f_{[a,m_1]\times[c,m_2]}\right)+R^2_{CC_5GL_3}\left(f_{[m_1,b]\times[c,m_2]}\right)+R^2_{CC_5GL_3}\left(f_{[m_1,b]\times[m_2,d]}\right)$ $+R^2_{CC_5GL_3}\left(f_{[a,m_1]\times[m_2,d]}\right)$ is compared with $R^2_{CC_5GL_3}\left(f_{[a,b]\times[c,d]}\right)$ to estimate the error in $R^2_{CC_5GL_3}\left(f_{[a,m_1]\times[c,m_2]}\right)+R^2_{CC_5GL_3}\left(f_{[m_1,b]\times[c,m_2]}\right)+R^2_{CC_5GL_3}\left(f_{[m_1,b]\times[m_2,d]}\right)+R^2_{CC_5GL_3}\left(f_{[a,m_1]\times[m_2,d]}\right)$.

Step 4: If the $|\text{estimated error}|\le\frac{\varepsilon}{2}(\text{termination criterion})$ then $R^2_{CC_5GL_3}\left(f_{[a,m_1]\times[c,m_2]}\right)$ $+R^2_{CC_5GL_3}\left(f_{[m_1,b]\times[c,m_2]}\right)+R^2_{CC_5GL_3}\left(f_{[m_1,b]\times[m_2,d]}\right)+R^2_{CC_5GL_3}\left(f_{[a,m_1]\times[m_2,d]}\right)$ is accepted as an approximation to the surface integral $I(f)=\int_a^b\int_c^d f(x,y)\,dxdy$. Otherwise the same procedure is applied to each of the four rectangles allowing each piece of rectangles a tolerance $\frac{\varepsilon}{2}$.

If the termination criterion is not satisfied on one or more of the rectangles, then those rectangles must be further split into four sub-rectangles and the entire process is repeated. When the process stops, the addition of all accepted values yields the desired approximate value $Q(f)$ to the surface integral $I(f)$ such that $|Q(f)-I(f)|\le\varepsilon$.

N.B: In this algorithm we can use any cubature rule to evaluate surface integrals in adaptive scheme.

5. Numerical verification

For the numerical verification of the mixed cubature rule $\left(R^2_{CC_5GL_3}(f)\right)$, the following surface integrals are considered

Table 1: Comparative study of the cubature/ mixed Cubature rule for approximation of some surface integrals in non-adaptive scheme

Integrals	Exact Value $(I(f))$	Approximate Value $(Q(f))$		
		$R^2_{CC_5}(f)$	$R^2_{GL_3}(f)$	$R^2_{CC_5GL_3}(f)$
$\int_{-1}^{1}\int_{-1}^{1} e^{x+y}dxdy$	5.524391382167	**5.5242**644124	**5.5248**367316	**5.52439**35083
$\int_0^1\int_0^1 \frac{x}{(xy+1)^2}dxdy$	0.3068528194	**0.30685**44528	**0.30685**69362	**0.306852**67902
$\int_0^2\int_0^1 \sin\left(\sqrt{x^3+y^3}\right)dxdy$	1.381737122	**1.3811**660279	**1.38**0779084	**1.3814**424161
$\int_{-1}^{1}\int_{-1}^{1} e^{-\left(x^2+y^2\right)}dxdy$	2.2309851414041	**2.23**80657547	**2.2**460405304	**2.23**23694866

Table 2: Comparative study of two mixed cubature rules for approximation of surface integrals (as same as table-1) in non-adaptive scheme

Integrals	Exact Value $(I(f))$	Approximate Value $(Q(f))$	
		$R^2_{S_3GL_2}(f)$	$R^2_{CC_5GL_3}(f)$
$\int_{-1}^{1}\int_{-1}^{1} e^{x+y}dxdy$	5.524391382167	**5.524**654155705	**5.52439**35083
$\int_0^1\int_0^1 \frac{x}{(xy+1)^2}dxdy$	0.3068528194	**0.3068**460735304	**0.306852**67902
$\int_0^2\int_0^1 \sin\left(\sqrt{x^3+y^3}\right)dxdy$	1.381737122	**1.38**267107252405	**1.3814**424161
$\int_{-1}^{1}\int_{-1}^{1} e^{-\left(x^2+y^2\right)}dxdy$	2.2309851414041	**2.2**2897496086442	**2.23**23694866

Table 3: Comparative study of the cubature/ mixed cubature rule for approximation of surface integrals (as same as table-1) using adaptive scheme

Integrals	Approximate Value $(Q(f))$					
	$R^2_{CC_5}(f)$	# Steps	$R^2_{GL_3}(f)$	# Steps	$R^2_{CC_5GL_3}(f)$	# Steps
$\int_{-1}^{1}\int_{-1}^{1} e^{x+y}dxdy$	**5.5243913**65409	09	**5.5243913**74787	17	**5.524391382**204	**05**
$\int_{0}^{1}\int_{0}^{1} \frac{x}{(xy+1)^2}dxdy$	**0.30685282**0224	05	**0.306852819**844	13	**0.30685281**8188	**01**
$\int_{0}^{2}\int_{0}^{1} \sin\left(\sqrt{x^3+y^3}\right)dxdy$	**1.3817371**45515	29	**1.381737129**753	37	**1.381737**082031	**13**
$\int_{-1}^{1}\int_{-1}^{1} e^{-(x^2+y^2)}dxdy$	**2.23098514**039	21	**2.2309851**39001	21	**2.23098514**139	**21**

Table 4: Comparative study of two mixed cubature rules for approximation of surface integrals (as same as table-1) using adaptive scheme

Integrals	Approximate Value $(Q(f))$			
	$R^2_{S_3GL_2}(f)$	# Steps	$R^2_{CC_5GL_3}(f)$	# Steps
$\int_{-1}^{1}\int_{-1}^{1} e^{x+y}dxdy$	**5.52439138**63996	17	**5.524391382**204	**05**
$\int_{0}^{1}\int_{0}^{1} \frac{x}{(xy+1)^2}dxdy$	**0.30685281**89525	05	**0.30685281**8188	**01**
$\int_{0}^{2}\int_{0}^{1} \sin\left(\sqrt{x^3+y^3}\right)dxdy$	**1.381737122**69338	29	**1.381737**082031	**13**
$\int_{-1}^{1}\int_{-1}^{1} e^{-(x^2+y^2)}dxdy$	**2.23098514**273695	21	**2.23098514**139	**21**

Here the prescribed tolerance **ε=0.000001**
Steps: Number of steps
All the computations are done using ‘C’ program.

Observation

In Table-1&2, we observe that the results of the new mixed cubature rule $\left(R^2_{CC_5GL_3}(f)\right)$ are more accurate than its constituent rules. On comparison, also we see that our mixed cubature rule returns much better results than the mixed cubature of Simpson's $\frac{1}{3}$ rd rule and Gauss-Legendre 2-point rule $\left(R^2_{S_3GL_2}(f)\right)$.

In Table-3&4, we evaluate the same test integrals given in the table-1 using the new mixed cubature rule in adaptive scheme. We see that in evaluation of each test integral the number of steps used to achieve the prescribed tolerance declines in case of the mixed cubature rule $\left(R^2_{CC_5GL_3}(f)\right)$.

6. Conclusion

Basing on the observation we conclude, the mixed cubature rule $\left(R^2_{CC_5GL_3}(f)\right)$ is not only effective in comparison to the corresponding constituent cubature rules $R^2_{CC_5}(f)$, $R^2_{GL_3}(f)$ and the mixed cubature rule $\left(R^2_{S_3GL_2}(f)\right)$ in non-adaptive environment but also it is much more potential and impressive in adaptive environment so far the number of steps and accuracy are concerned. Therefore, in scientific computations one must prefer the mixed cubature rule to its constituent rules and other mixed cubature rule in adaptive mode.

Acknowledgement. We are very much thankful to the reviewers who in spite of their pressing engagements took keen interest in reviewing our paper. We humbly accept their valuable suggestions to improve the quality of the paper and work out accordingly.

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