On the Euler’s Form of an odd Perfect Number

Balchandar Reddy Sangam

Department of Mathematics
Birla Institute of Technology & Science, Pilani, Hyderabad, India
Email: sbr18334@gmail.com

Received 1 August 2020; accepted 20 August 2020

Abstract. Euler has proved that an odd perfect number, if exists, must be of the form \( p^\alpha q_1^{2\beta_1}q_2^{2\beta_2}...q_r^{2\beta_r}, \) \( p \equiv \alpha \equiv 1 \mod 4. \) In this article, we show:
(i) An alternative proof to the Euler’s form of odd perfect numbers.
(ii) An odd number of the form: \( p^\alpha q^{2\beta}, \) \( p \equiv \alpha \equiv 1 \mod 4 \) cannot be perfect.

Keywords: Odd perfect numbers, Euler’s form.

AMS Mathematics Subject Classification (2010): 11A25, 11A41

1. Introduction
A natural number \( N \) is said to be perfect if \( N \) equals the sum of all its proper divisors. As of May 2020, fifty-one perfect numbers have been found, all of which are even. All even perfect numbers have one-one correspondence with Mersenne primes. Euclid proved that if \( 2^{p-1} \) is prime, then its corresponding perfect number is given as \( 2^{p-1}(2^p - 1). \) The existence of an odd perfect number still remains to be an open question. But many forms of an odd perfect number were given by the researchers.

Recent work by Hare [1] states that an odd perfect number if exists must have 75 or more factors. In 1973, Pomerance [8] proved that an odd perfect number is divisible by at least 7 distinct prime factors. In 1979, Chein [7] improved the number of distinct prime factors of an odd perfect number to 8. In 2003, Jenkins [5] showed that the odd perfect numbers have a prime factor exceeding \( 10^7. \) Nielsen [6] provided an upper bound for odd perfect numbers to be \( 2^{4k} \) with \( k \) distinct prime factors. There have been various other advancements [3-4] on perfect numbers in the last decade. Euler [2] has given the form of an odd perfect number as \( p^\alpha q_1^{2\beta_1}q_2^{2\beta_2}...q_r^{2\beta_r}, \) \( p \equiv \alpha \equiv 1 \mod 4. \) In this article, we provide an alternative proof to the Euler’s form of odd perfect numbers.

2. Results
Theorem 2.1. An odd number with each prime factor occurring only once in its prime factorisation cannot be perfect.

Proof: Consider an odd number \( N, \) where \( N = \prod_{i=1}^{n} p_i \) with \( p_1, p_2, p_3, ..., p_n \) as its prime factors. As all the prime factors of \( N \) are odd, \( p_i \) can be represented as \( 2k_i + 1. \) All prime
Balchandar Reddy Sangam

factors to \( N \) will be represented in the similar fashion. Throughout this article the sum of all proper factors of \( N \) is given as \( \sigma(N) \).

For example, let’s assume that \( N \) has only three factors \( p_1, p_2, p_3 \), then

\[
N = (2k_1 + 1)(2k_2 + 1)(2k_3 + 1).
\]

\[
\sigma(N) = 1 + (2k_1 + 1) + (2k_2 + 1) + (2k_3 + 1) + (2k_1 + 1)(2k_2 + 1) + (2k_2 + 1)(2k_3 + 1) + (2k_1 + 1)(2k_3 + 1).
\]

\[
\sigma(N) = 4k_1k_2 + 4k_2k_3 + 4k_1k_3 + 6k_1 + 6k_2 + 6k_3 + 7.
\]

\[
N = 8k_1k_2k_3 + 4k_1k_2 + 4k_2k_3 + 4k_1k_3 + 2k_1 + 2k_2 + 2k_3 + 1.
\]

(1)

(2)

For the number to be perfect, \( \sigma(N) \) and \( N \) both should be equal.

Subtracting (2) from (1), we obtain

\[
\sigma(N) - N = 4k_1 + 4k_2 + 4k_3 - 8k_1k_2k_3 + 6.
\]

(3)

As \( \sigma(N) - N = 0 \), it can be inferred that for any values of \( k_1, k_2, k_3 \), (3) doesn’t stand true, which indicates that \( N \) cannot be perfect. The same is extended to \( N \) with \( n \) prime factors.

Consider an odd number \( N \) with \( n \) prime factors named \( p_1, p_2, p_3, ..., p_n \). \( \sigma(N) \) has a constant term which we represent as \( S_0 \), the factors with degree 1 are represented as \( S_1 \).

Similarly, the factors with degree \( m \) are represented as \( S_m \).

Therefore,

\[
\sigma(N) = S_0 + S_1 + S_2 + ... + S_{n-1}.
\]

(4)

For example, \( S_1 \) is given as \( p_1 + p_2 + p_3 + ... + p_n \), which in turn can be written as \( (2k_1 + 1) + (2k_2 + 1) + (2k_3 + 1) + ... + (2k_n + 1) \). The sum of higher degree terms in each of \( S_0, S_1, S_2, ..., S_{n-1} \) will be the sum of all non-higher degree terms of \( N \).

Let \( A = \text{Sum of } S_0, S_1, S_2, ..., S_{n-1} \) without their higher degree terms \( , B = \text{Higher degree terms of } N \).

\[
\sigma(N) - N = A - B.
\]

The number of factors that \( N \) has is \( 0 \pmod{4} \). So, the constant term in the expansion of \( \sigma(N) - N \) will be \( 2 \pmod{4} \). Here if it can be proved that all the terms other than the constant have the coefficient as \( 0 \pmod{4} \), we can conclude that \( N \) is not perfect.

The coefficient of a term with degree \( m \) appearing anywhere in \( S_0, S_1, S_2, ..., S_{n-1} \) will be \( 2^m \), which indicates that any term with degree 2 or more has the coefficient as \( 0 \pmod{4} \).

The coefficient of a term with degree 1 (among \( k_1, k_2, ..., k_n \) in \( S_m \) is given as,

\[
2 \frac{m!}{n!} \left( \frac{2n-1}{m-1} \right)
\]

Coefficient of a term with degree 1 in the expansion of \( \sigma(N) - N \) is

\[
2 \sum_{m=2}^{n-1} \frac{(n-1)}{m-1} 2(2^{n-1} - 2) \equiv 0 \pmod{4}
\]

As the coefficients of all the terms with degree 1 or more is \( 0 \pmod{4} \) and the constant term is \( 2 \pmod{4} \). Hence, we conclude that \( N \) with the given form, is not perfect.
On The Euler’s Form of an odd Perfect Number

Lemma 2.2. An odd number with either 0 (mod 4), 1 (mod 4) or 3 (mod 4) factors is not perfect.

Proof: Let us assume that an odd number has \( f \) factors. As stated in Theorem 2.1, the constant term \( c \) in the expansion of \( \sigma(N)-N \) depends on the number of factors of \( N \) whereas the coefficients of all the higher degree terms is 0 (mod 4).

From Theorem 2.1, we have \( c = f - 2 \). If \( f \equiv 2 \) (mod 4) then \( c \equiv 0 \) (mod 4), any other form of \( f \) would lead to \( c \) being either of 1 (mod 4), 2 (mod 4), 3 (mod 4), which contradicts with the form of odd perfect numbers. So, for an odd number to be perfect it should have 2 (mod 4) factors.

Lemma 2.3. The only form of an odd number with 2 (mod 4) factors is

\[ p^α q_1^{2β_1} q_2^{2β_2} \ldots q_r^{2β_r}, \alpha \equiv 1 \ (\text{mod} \ 4) \]

Proof: As we know,

\[ \sum_{k=1}^{\alpha} \frac{4n+2}{\alpha+1} \cdot 3^m + \frac{4n+2}{\alpha+1} \cdot 3^{α-1} + \left( \frac{4n+2}{\alpha+1} - 2 \right) 3^α \]

which is,

\[ \equiv 0 \ (\text{mod} \ 4) + 0 \ (\text{mod} \ 4) - 2 \cdot 3 \alpha \]

Hence, we conclude that \( p \) should be 1 (mod 4). The only form that an odd perfect number takes is \( p^α q_1^{2β_1} q_2^{2β_2} \ldots q_r^{2β_r}, p \equiv \alpha \equiv 1 \ (\text{mod} \ 4) \). This concludes the proof of Euler’s form of an odd perfect number.
Balchandar Reddy Sangam

**Theorem 2.5.** An odd number of the form: \( p^\alpha q^2, p \equiv \alpha \equiv 1 \pmod{4} \) cannot be perfect.

**Proof:** Let \( N = p^\alpha q^2 \).

\[
\sigma(N) = (1 + p + p^2 + \ldots + p^\alpha)(q^2 + q + 1) - N
\]
As \( \sigma(N) = N \),
\[
(1 + p + p^2 + \ldots + p^\alpha)(1 + q + q^2) = 2N
\]
\[
\left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^\alpha}\right) \left(1 + \frac{1}{q} + \frac{1}{q^2}\right) = 2 \quad (7)
\]
Considering (7) to be of the form,
\[
(1 + x)(1 + y) = 2
\]
Then, \( x = \frac{1-y}{1+y}, y \neq -1 \) and (7) can be represented as,
\[
\left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^\alpha}\right) = \frac{(1 - \frac{1}{q^2})}{(1 + \frac{1}{q^2})}
\]
\[
\frac{p^{\alpha-1} + p^{\alpha-2} + \ldots + 1}{p^\alpha} = \frac{q^2 - q - 1}{q^2 + q + 1} \quad (8)
\]
If \( \frac{a}{b} = \frac{c}{d} \); \( a, b \) are co-primes and \( c, d \) are co-primes then \( a = c \) and \( b = d \).

As \( p \) is odd, \( p^{\alpha-1} + p^{\alpha-2} + \ldots + 1 \) and \( p^\alpha \) are co-primes. \( q^2 - q - 1 \) and \( q^2 + q + 1 \) are co-primes as well. So, the only possible solution to (8) is
\[
p^{\alpha-1} + p^{\alpha-2} + \ldots + 1 = q^2 - q - 1 \quad (9)
\]
\[
p^\alpha = q^2 + q + 1 \quad (10)
\]
(9) can also be represented as,
\[
\frac{p^{\alpha-1}}{p-1} = q^2 - q - 1 \quad (11)
\]
Substituting (10) in (11), we have
\[
p - 1 = \frac{q^2 + q}{q^2 - q - 1} \quad (12)
\]
There is no positive integer \( q \) for which \( \frac{q^2 + q}{q^2 - q - 1} \) is an integer, which indicates that there is no \( p \) value that satisfies (12). Hence, we conclude that an odd number of the form \( p^\alpha q^2 \) is not perfect.

**Theorem 2.6.** An odd number of the form: \( p^\alpha q^{2\beta}, p \equiv \alpha \equiv 1 \pmod{4} \) is not perfect.

**Proof:** Let \( N = p^\alpha q^{2\beta} \). Rewriting (12), we have
\[
p - 1 = \frac{q^{2\beta} + q^{2\beta - 1} + \ldots + q}{q^{2\beta} - q^{2\beta - 1} - \ldots - q - 1} \quad (13)
\]
Let us consider that \( z = \frac{q^{2\beta} + q^{2\beta - 1} + \ldots + q}{q^{2\beta} - q^{2\beta - 1} - \ldots - q - 1} \)
\[
z = q^{a - \frac{q^a}{a - q^{a-1}}}, \text{where } a = q^{2\beta} - q^{2\beta - 1} + \ldots + 1
\]
\[
z = \frac{q^a}{a(q - 1) + a - q^{a-1}} \text{ as } q^{2\beta} = a(q - 1) + 1
\]
\[
z = \frac{q^a}{a(q - 2) + 1} \quad (14)
\]
z has negative gradient at all \( q > 0 \), so it has a maximum value at \( q = 3 \).
At \( q = 3 \), \( z = \frac{3a}{a + 1} \). Hence, \( z \in (1, 3) \).

The only integer value in this region that \( z \) can take is 2. Substituting \( z = 2 \) in (14), we get
On The Euler’s Form of an odd Perfect Number

$$\frac{qa}{a(q-2)+1} = 2; qa = 2(a(q - 2) + 1)$$

$$qa - 4a + 2 = 0$$

$$a(q - 4) = -2$$

(15)

As \(a\) is positive, For (15) to be true \(q < 4\), The only odd prime which is less than 4 is 3.

So, \(q = 3\). From (14) it is clear that \(z\) is not an integer, when \(q = 3\), as \(z = \frac{3a}{a+1}\).

We can deduce that there is no prime \(q\) for which (13) stands true. Hence, we conclude that an odd number of the form \(p^\alpha q^{2\beta}\) is not perfect.

3. Conclusion and future scope

This work can be extended in various directions such as, how the factors with even powers behave with respect to \(\alpha\). The main idea behind this article is to prove that odd perfect numbers do not exist. So, the proof of Theorem 2.6, could certainly help the fellow researchers to proceed in that direction.

Acknowledgements. I would like to thank the referees for their comments and suggestions on the manuscript. I also would like to express my deep gratitude to all the authors, professors, and experts for their contributions.

REFERENCES