Annals of Pure and Applied Mathematics
Vol. 22, No. 1, 2020, 51-55
ISSN: 2279-087X (P), 2279-0888(online)
Published on 21 August 2020
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v22n1a07685

Annals of
Pure and Applied Mathematics

On the Euler's Form of an odd Perfect Number

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Received 1 August 2020; accepted 20 August 2020

Abstract. Euler has proved that an odd perfect number, if exists, must be of the form $p^{\alpha}{q_1}^{2\beta_1}{q_2}^{2\beta_2}...{q_r}^{2\beta_r}$, $p \equiv \alpha \equiv 1$ (mod 4). In this article, we show:
(i) An alternative proof to the Euler's form of odd perfect numbers.
(ii) An odd number of the form: $p^{\alpha}q^{2\beta}$, $p \equiv \alpha \equiv 1$ (mod 4) cannot be perfect.

Keywords: Odd perfect numbers, Euler's form.

AMS Mathematics Subject Classification (2010): 11A25, 11A41

1. Introduction

A natural number *N* is said to be perfect if *N* equals the sum of all its proper divisors. As of May 2020, fifty-one perfect numbers have been found, all of which are even. All even perfect numbers have one-one correspondence with Mersenne primes. Euclid proved that if 2^{p-1} is prime, then its corresponding perfect number is given as $2^{p-1}(2^p - 1)$. The existence of an odd perfect number still remains to be an open question. But many forms of an odd perfect number were given by the researchers.

Recent work by Hare [1] states that an odd perfect number if exists must have 75 or more factors. In 1973, Pomerance [8] proved that an odd perfect number is divisible by at least 7 distinct prime factors. In 1979, Chein [7] improved the number of distinct prime factors of an odd perfect number to 8. In 2003, Jenkins [5] showed that the odd perfect numbers have a prime factor exceeding 10^7. Nielsen [6] provided an upper bound for odd perfect numbers to be 2^{4^k} with k distinct prime factors. There have been various other advancements [3-4] on perfect numbers in the last decade. Euler [2] has given the form of an odd perfect number as $p^{\alpha}{q_1}^{2\beta_1}{q_2}^{2\beta_2}...{q_r}^{2\beta_r}$,$p \equiv \alpha \equiv 1$ (mod 4). In this article, we provide an alternative proof to the Euler's form of odd perfect numbers.

2. Results

Theorem 2.1. An odd number with each prime factor occurring only once in its prime factorisation cannot be perfect.

Proof: Consider an odd number *N*, where $N = \prod_{i=1}^{n} p_i$ with $p_1, p_2, p_3, \ldots, p_n$ as its prime factors. As all the prime factors of *N* are odd, p_i can be represented as $2k_i + 1$. All prime factors to N will be represented in the similar fashion. Throughout this article the sum of all proper factors of N is given as σ(N).

For example, let's assume that N has only three factors p_1, p_2, p_3, then

$$N = (2k_1+1)(2k_2+1)(2k_3+1).$$

$$\sigma(N) = 1 + (2k_1+1) + (2k_2+1) + (2k_3+1) + (2k_1+1)(2k_2+1) + (2k_2+1)(2k_3+1) + (2k_1+1)(2k_3+1).$$

$$\sigma(N) = 4k_1k_2 + 4k_2k_3 + 4k_1k_3 + 6k_1 + 6k_2 + 6k_3 + 7. \quad (1)$$

$$N = 8k_1k_2k_3 + 4k_1k_2 + 4k_2k_3 + 4k_1k_3 + 2k_1 + 2k_2 + 2k_3 + 1. \quad (2)$$

For the number to be perfect, σ(N) and N both should be equal.
Subtracting (2) from (1), we obtain

$$\sigma(N) - N = 4k_1+4k_2+4k_3-8k_1k_2k_3 + 6. \quad (3)$$

As σ(N)−N = 0, it can be inferred that for any values of k_1, k_2, k_3, (3) doesn't stand true, which indicates that N cannot be perfect. The same is extended to N with n prime factors.

Consider an odd number N with n prime factors named $p_1, p_2, p_3, \ldots, p_n$. σ(N) has a constant term which we represent as S_0, the factors with degree 1 are represented as S_1. Similarly, the factors with degree m are represented as S_m.
Therefore,

$$\sigma(N) = S_0 + S_1 + S_2 + \ldots + S_{n-1}. \quad (4)$$

For example, S_1 is given as $p_1+p_2+p_3+\ldots+p_n$, which in turn can be written as $(2k_1+1) + (2k_2+1) + (2k_3+1) + \cdots\ldots+(2k_n+1)$. The sum of higher degree terms in each of $S_0, S_1, S_2, \ldots, S_{n-1}$ will be the sum of all non-higher degree terms of N.
Let A = Sum of $S_0, S_1, S_2, \ldots, S_{n-1}$ without their higher degree terms , B = Higher degree terms of N. So,

$$\sigma(N)\text{-}N = A - B.$$

The number of factors that N has is 0 (mod 4). So, the constant term in the expansion of σ(N)-N will be 2 (mod 4). Here if it can be proved that all the terms other than the constant have the coefficient as 0 (mod 4), we can conclude that N is not perfect.

The coefficient of a term with degree m appearing anywhere in $S_0, S_1, S_2, \ldots, S_{n-1}$ will be 2^m, which indicates that any term with degree 2 or more has the coefficient as 0 (mod 4).
The coefficient of a term with degree 1 (among $k_1, k_2, \ldots, k_n$) in S_m is given as,

$$2\frac{m\binom{n}{m}}{n}$$

$$\binom{2n-1}{m-1}$$

Coefficient of a term with degree 1 in the expansion of σ(N)-N is

$$2\sum_{m=2}^{n-1}\binom{n-1}{m-1}$$

$$2(2^{n-1}-2) \equiv 0 \pmod 4$$

As the coefficients of all the terms with degree 1 or more is 0 (mod 4) and the constant term is 2 (mod 4). Hence, we conclude that N with the given form, is not perfect.

Lemma 2.2. An odd number with either 0 (mod 4), 1 (mod 4) or 3 (mod 4) factors is not perfect.
Proof: Let us assume that an odd number has f factors. As stated in Theorem 2.1, the constant term c in the expansion of σ(N)-N depends on the number of factors of N whereas the coefficients of all the higher degree terms is 0 (mod 4).
From Theorem 2.1, we have $c = f - 2$, If $f \equiv 2$ (mod 4) then $c \equiv 0$ (mod 4), any other form of f would lead to c being either of 1 (mod 4), 2 (mod 4), 3 (mod 4), which contradicts with the form of odd perfect numbers. So, for an odd number to be perfect it should have 2 (mod 4) factors.

Lemma 2.3. The only form of an odd number with 2 (mod 4) factors is
$p^{\alpha} {q_1}^{2\beta_1} {q_2}^{2\beta_2} ... {q_r}^{2\beta_r}, \alpha \equiv 1$ (mod 4)
Proof: As we know,

$$f = (p_1 + 1)(p_2 + 1)(p_3 + 1)....(p_n + 1). \tag{5}$$

From Lemma 2.2, it is understood that an odd perfect number has 2 (mod 4) factors.

$$f \equiv 2 \pmod 4. \tag{6}$$

From (5) & (6), we have
(1) Exactly one of the terms among $(p_1 + 1)(p_2 + 1)(p_3 + 1)...(p_n + 1)$ is even.
(2) Furthermore, the even power cannot be 0 (mod 4), as that would make f to be 0 (mod 4).
(3) Therefore, exactly one power from $p_1, p_2, p_3, ..., p_n$ will be odd. From (2) it can be inferred that the odd power (α) is 1 (mod 4).

Here we conclude that exactly one power(α) among the powers of all prime factors should be odd and additionally it should be 1 (mod 4).

Theorem 2.4. Odd number N from Lemma 2.3 cannot be perfect if $p \equiv 3$ (mod 4), which makes the odd perfect number to be $p^{\alpha} {q_1}^{2\beta_1} {q_2}^{2\beta_2} ... {q_r}^{2\beta_r}, p \equiv \alpha \equiv 1$ (mod 4).
Proof: Let us assume that $p \equiv 3$ (mod 4), each power of p ranging from 0 to α occurs an equal number of times in the expansion of σ(N)+N. Similarly in the expansion of σ(N)-N except the highest power(α), all the other power occurs an equal number of times, which is $\frac{4n+2}{\alpha+1}$ times, whereas the higher power of p occurs $\frac{4n+2}{\alpha+1} - 2$ times.
So, the constant term in the expansion of σ(N)-N becomes,

$$\sum_{m=0}^{\alpha-1} \frac{4n+2}{\alpha+1} 3^m + \left(\frac{4n+2}{\alpha+1} - 2\right) 3^{\alpha}$$

$$\sum_{m=0}^{\alpha-2} \frac{4n+2}{\alpha+1} 3^m + \frac{4n+2}{\alpha+1} 3^{\alpha-1} + \left(\frac{4n+2}{\alpha+1} - 2\right) 3^{\alpha}$$

which is,

$$\equiv 0 \pmod 4 + 0 \pmod 4 - 2 \,.\, 3\, \alpha$$
$$\equiv 2 \pmod 4$$

Hence, we conclude that p should be 1 (mod 4). The only form that an odd perfect number takes is $p^{\alpha} {q_1}^{2\beta_1} {q_2}^{2\beta_2} ... {q_r}^{2\beta_r}, p \equiv \alpha \equiv 1$(mod 4). This concludes the proof of Euler's form of an odd perfect number.

Theorem 2.5. An odd number of the form: $p^{\alpha}q^{2}, p \equiv \alpha \equiv 1 \pmod 4$ cannot be perfect.
Proof: Let $N = p^{\alpha}q^{2}$.

$$\sigma(N) = (1 + p + p^2 + \ldots + p^{\alpha})(q^2 + q + 1) - N$$

As σ(N) = N,

$$(1 + p + p^2 + \ldots + p^{\alpha})(1 + q + q^2) = 2N$$

$$(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^{\alpha}})(1 + \frac{1}{q} + \frac{1}{q^2}) = 2 \tag{7}$$

Considering (7) to be of the form,

$$(1 + x)(1 + y) = 2$$

Then, $x = \frac{1-y}{1+y}; y \neq -1$ and (7) can be represented as,

$$(\frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^{\alpha}}) = \frac{(1-\frac{1}{q}-\frac{1}{q^2})}{(1+\frac{1}{q}+\frac{1}{q^2})}$$

$$\frac{p^{\alpha-1}+p^{\alpha-2}+\ldots+1}{p^{\alpha}} = \frac{q^2-q-1}{q^2+q+1} \tag{8}$$

If $\frac{a}{b} = \frac{c}{d}$; a, b are co-primes and c, d are co-primes then $a = c$ and $b = d$.
As p is odd, $p^{\alpha-1} + p^{\alpha-2} + \ldots + 1$ and p^{α} are co-primes. $q^2 - q - 1$ and $q^2 + q + 1$ are co-primes as well.So, the only possible solution to (8) is

$$p^{\alpha-1} + p^{\alpha-2} + \ldots + 1 = q^2 - q - 1 \tag{9}$$

$$p^{\alpha} = q^2 + q + 1 \tag{10}$$

(9) can also be represented as,

$$\frac{p^{\alpha}-1}{p-1} = q^2 - q - 1 \tag{11}$$

Substituting (10) in (11), we have

$$p - 1 = \frac{q^2+q}{q^2-q-1} \tag{12}$$

There is no positive integer q for which $\frac{q^2+q}{q^2-q-1}$ is an integer, which indicates that there is no p value that satisfies (12). Hence, we conclude that an odd number of the form $p^{\alpha}q^{2}$ is not perfect.

Theorem 2.6. An odd number of the form: $p^{\alpha}q^{2\beta}, p \equiv \alpha \equiv 1 \pmod 4$is not perfect.
Proof: Let $N = p^{\alpha}q^{2\beta}$, Rewriting (12), we have

$$p - 1 = \frac{q^{2n}+q^{2n-1}+\ldots+q}{q^{2n}-q^{2n-1}-\ldots-q-1} \tag{13}$$

Let us consider that $z = \frac{q^{2n}+q^{2n-1}+\ldots+q}{q^{2n}-q^{2n-1}-\ldots-q-1}$.

$$z = \frac{qa}{q^{2n}-a}, \text{ where } a = q^{2n-1} + q^{2n-2} + \ldots + 1$$

$$z = \frac{qa}{a(q-1)+1-a}, \text{ as } q^{2n} = a(q-1) + 1$$

$$z = \frac{qa}{a(q-2)+1}. \tag{14}$$

z has negative gradient at all $q > 0$, so it has a maximum value at $q = 3$.
At $q = 3$, $z = \frac{3a}{a+1}$. Hence, $z \in (1,3)$.
The only integer value in this region that z can take is 2. Substituting $z = 2$ in (14), we get

$$\frac{qa}{a(q-2)+1} = 2;\ qa = 2(a(q-2)+1)$$
$$qa - 4a + 2 = 0$$
$$a(q-4) = -2 \qquad (15)$$

As a is positive, For (15) to be true $q < 4$, The only odd prime which is less than 4 is 3. So, q = 3.From (14) it is clear that z is not an integer, when q = 3, as $z = \frac{3a}{a+1}$.
We can deduce that there is no prime q for which (13) stands true. Hence, we conclude that an odd number of the form $p^{\alpha}q^{2\beta}$ is not perfect.

3. Conclusion and future scope

This work can be extended in various directions such as, how the factors with even powers behave with respect to α. The main idea behind this article is to prove that odd perfect numbers do not exist. So, the proof of Theorem 2.6, could certainly help the fellow researchers to proceed in that direction.

Acknowledgements. I would like to thank the referees for their comments and suggestions on the manuscript. I also would like to express my deep gratitude to all the authors, professors, and experts for their contributions.

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