Annals of Pure and Applied Mathematics
Vol. 22, No. 1, 2020, 57-64
ISSN: 2279-087X (P), 2279-0888(online)
Published on 26 August 2020
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v22n1a08690

Annals of
Pure and Applied
Mathematics

Common Fixed Point Theorems for Six Self-Maps in G-Metric Spaces

V.Nagaraju

Department of Mathematics, University College of Science
Osmania University, Hyderabad, Telangana, India
E-mail: viswanag2007@gmail.com

Received 25 May 2020; accepted 2 August 2020

Abstract. In this paper, we define generalized weakly contractive mappings and prove two common fixed point theorems for six self maps in G-metric spaces. Using the notions of common limit in the range property and weakly compatibility, the first theorem is proved whereas occasionally weakly compatibility is used in the second theorem. We also deduce some common fixed point theorems for four self maps. Our results improve and extend some metric fixed point results in the literature.

Keywords: G-metric space, Fixed point, Weakly compatible mappings, Common limit in the range property, Occasionally weakly compatible mappings

AMS Mathematics Subject Classification (2010): 47H10, 54H25

1. Introduction

In an attempt to generalize the notion of a metric space, Gahler and Dhage proposed different generalizations and proved many fixed point theorems for such generalizations. Later, Mustafa and Sims proved that most of the results claimed by Dhage are invalid**.** In 2006, Mustafa and Sims [5] introduced G-metric space by generalizing metric space. After that, several authors studied many fixed point and common fixed point results for self-mappings in G-metric spaces under certain contractive conditions [5,6,7].

On the other hand, in 2011, Sintunavarat et al. [8] introduced the notion of common limit in the range of F (CLR_F) property for a pair of self mappings in metric spaces. Afterwards, several common fixed point theorems were studied by many authors under this notion. Recently, Karapinar et al. [3] extended the concept of common limit range property to two pairs of self mappings in symmetric spaces. For more results on fixed points, we refer to [9,10,11].

2. Preliminaries

In this section, we present some definitions and results which will be used in the sequel.

Definition 2.1. [5] Let X be a non empty set and let $G: X \times X \times X \to [0,\infty)$ be a function satisfying the following properties:

$G(x,y,z) = 0$ if $x = y = z$.

$G(x,x,y) > 0$ for all $x,y \in X$, with $x \neq y$.

$G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$, with $y \neq z$.
$G(x,y,z) = G(x,z,y) = G(y,z,x) =$..., (symmetry in all three variables)
$G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$
(rectangular inequality)

Then the function G is called a G-metric on X and the pair (X,G) is called a G-metric space.

Example 2.1. [5] Let (X,d) be a usual metric space. Then (X,G) is G-metric space, where $G(x,y,z) = d(x,y) + d(y,z) + d(x,z)$, for all $x,y,z \in X$.

Definition 2.2. [6] A G-metric space (X, G) is said to be symmetric if $G(x,y,y) = G(y,x,x)$ for all x, y ∈ X.

Proposition 2.3. [6] Every G-metric space (X,G) defines a metric space (X,d_G), where d_G defined by

$$d_G(x,y) = G(x,y,y) + G(y,x,x) \text{ for all } x,y \in X.$$

Proposition 2.4. [5] Let (X,G) be a G-metric space. Then for any $x,y,z,a \in X$, the following hold.

$G(x,y,z) = 0$ then $x = y = z$.
$G(x,y,z) \leq G(x,x,y) + G(x,x,z)$.
$G(x,y,y) \leq 2\, G(y,x,x)$.
$G(x,y,z) \leq G(x,a,z) + G(a,y,z)$.
$G(x,y,z) \leq \frac{2}{3}\{G(x,a,a) + G(y,a,a) + G(z,a,a)\}$.

Definition 2.5. [2] A mapping $\mu: [0,\infty) \to [0,\infty)$ is called an altering distance if μ is continuous, non-decreasing and $\mu(t) = 0$ if and only if $t = 0$.

Definition 2.6. [1] Two self-mappings E and F of a metric space (X, d) are said to be weakly compatible if Efu = FEu whenever Eu=Fu, for some $u \in X$.

Definition 2.7. [1] Two self-mappings E and F are said be occasionally weakly compatible (owc) if EFu = FEu for some coincidence point u of E and F.

Remark 2.8. Every pair of weakly compatible mappings is occasionally weakly compatible but the converse is not true in general.

Definition 2.9. [8] Two self-maps E and F of a metric space (X, d) are said to satisfy the common limit in the range of F property, denoted by CLR_F property, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ex_n = \lim_{n\to\infty} Fx_n = Fu$ for some $u \in X$.

Definition 2.10. [3] Two pairs (A,E) and (f,g) of self-maps of a symmetric space (X, d) are said to satisfy the common limit range property with respect to the mappings E and g, denoted by $CLR_{(E,g)}$ property, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Ex_n = \lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = t$ with t = Eu = gv for some $t, u, v \in X$.

Remark 2.11. If A = f and E = g, then the above definition reduces to (CLR_E) property [8].

3. Main results

Let $C = \left\{ \begin{array}{l} \varphi / \varphi : [0,\infty) \to [0,\infty) \text{is lower semi continuous and non-decreasing} \\ \text{function such that } \varphi(t) = 0 \text{ iff } t = 0 \end{array} \right\}$.

Let P, Q, R, f, g and h be six self maps of a G-metric space (X, G) such that

$$\mu(G(Px, Qy, Qz)) \le \mu(L(x, y, z)) - \varphi(L(x, y, z)) \text{ for all } x, y, z \in X \quad (1)$$

where μ is an altering distance function, $\varphi \in C$ and

$$L(x, y, z) = \max\{G(Rfx, ghy, ghz), G(Rfx, Px, Px), G(ghy, Qy, Qz), \tfrac{1}{3}\{G(Rfx, Qy, Qz) + G(ghy, Px, Px)\}$$

Theorem 3.1. Let P, Q, R, f, g and h be six self maps of a symmetric G-metric space (X, G) satisfy (1) and also, if

3.1.1. (P,Rf) and (Q,gh) satisfy $CLR_{(Rf,gh)}$ propery and

3.1.2. (P,Rf) and (Q,gh) are weakly compatible, then the mappings P, Q, Rf and gh have a unique common fixed point in X. Further P, Q, R, f, g and h have a unique common fixed point in X provided the pairs of mappings (R,f), (P,R), (P,f), (g,h), (Q,g) and (Q,h) are commuting.

Proof: Since (P,Rf) and (Q,gh) satisfy $CLR_{(Rf,gh)}$ property, we can find two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Rfx_n = \lim_{n\to\infty} Qy_n = \lim_{n\to\infty} ghy_n = t$ with t = Rfu = ghv for some $t, u, v \in X$.

We first claim that Rfu = Pu.

From (1) with $x = u, y = z = y_n$, we get

$$\mu(G(Pu, Qy_n, Qy_n)) \le \mu(L(u, y_n, y_n)) - \varphi(L(u, y_n, y_n)) \quad (2)$$

where

$$L((u, y_n, y_n)) =$$

$$\max\{G(Rfu, ghy_n, ghy_n), G(Rfu, Pu, Pu), G(ghy_n, Qy_n, Qy_n), \tfrac{1}{3}\left\{\begin{array}{l} G(Rfu, Qy_n, Qy_n) \\ + \quad G(ghy_n, Pu, Pu) \end{array}\right\}\}$$

$$\text{So } \lim_{n\to\infty} L(u, y_n, y_n) = \max\{G(t,t,t), G(t, Pu, Pu), G(t,t,t), \tfrac{1}{3}\left\{\begin{array}{l} G(t,t,t) \\ + \quad G(t, Pu, Pu) \end{array}\right\}\}$$

$$= \max\{0, G(t, Pu, Pu), 0, \tfrac{1}{3}\{G(t, Pu, Pu)\}\}$$

$$= G(Pu, t, t), \text{ since G is symmetric.}$$

On letting $n \to \infty$ in (2),

$\mu(G(Pu, t, t)) \le \mu(G(Pu, t, t)) - \varphi(G(Pu, t, t))$ which implies that $\varphi(G(Pu, t, t)) = 0$ and so $G(Pu, t, t) = 0$, that is, $Pu = t = Rfu$, showing that u is a coincidence point of P and Rf.

Since (P, Rf) is weakly compatible, we have $P(Rf)u = (Rf)Pu$ and so $Pt = Rft$.

Claim : Qv = ghv.

From (1) with $x = x_n, y = z =$ v, we get that

$$\mu(G(Px_n, Qv, Qv)) \leq \mu(L(x_n, v, v)) - \varphi(L(x_n, v, v)) \qquad (3)$$

where

$L((x_n, v, v)) =$

$\max\{G(Rfx_n, ghv, ghv), G(Rfx_n, Px_n, Px_n), G(ghv, Qv, Qv), \frac{1}{3}\left\{\begin{matrix} G(Rfx_n, Qv, Qv) \\ + \quad G(ghv, Px_n, Px_n) \end{matrix}\right\}\}$

So $\lim_{n\to\infty} L((x_n, v, v)) = \max\{G(t,t,t), G(t,t,t), G(t, Qv, Qv), \frac{1}{3}\left\{\begin{matrix} G(t, Qv, Qv) \\ + \quad G(t,t,t) \end{matrix}\right\}\}$

$= \max\{0, 0, G(t, Qv, Qv), \frac{1}{3}\left\{\begin{matrix} G(t, Qv, Qv) \\ + \quad 0 \end{matrix}\right\}\}$

$= G(t, Qv, Qv).$

Taking limit as $n \to \infty$ in (3),

$\mu(G(t, Qv, Qv)) \leq \mu(G(t, Qv, Qv)) - \varphi(G(t, Qv, Qv))$ which implies that $\varphi(G(t, Qv, Qv)) = 0$ and so $G(t, Qv, Qv) = 0$, that is, $Qv = t = ghv$, showing that v is a coincidence point of Q and gh.

Since (Q, gh) is weakly compatible, we have $Q(gh)v = (gh)Qv$ and so $Qt = ght$

Claim : $Pt = Rft = t$.

From (1) with x = t, y = z = v, we have

$\mu\left(G(Pt, Qv, Qv)) \leq \mu(L(t, v, v)) - \varphi(L(t, v, v)\right)$ or

$\mu\left(G(Pt, t, t)) \leq \mu(L(t, v, v)) - \varphi(L(t, v, v)\right)$

where

$L(t, v, v) =$

$\max\{G(Rft, ghv, ghv), G(Rft, Pt, Pt), G(ghv, Qv, Qv), \frac{1}{3}\{G(Rft, Qv, Qv) + G(ghv, Pt, Pt)\}$

$= \max\{G(Pt, t, t), G(Pt, Pt, Pt), G(t, t, t), \frac{1}{3}\{G(Pt, t, t) + G(t, Pt, Pt)\}\}$

$= \max\{G(Pt, t, t), 0, 0, \frac{2}{3}\{G(Pt, t, t)\}\}$, since G is symmetric.

$= G(Pt, t, t)$

Hence, we get,

$\mu(G(Pt, t, t)) \leq \mu(G(Pt, t, t)) - \varphi(G(Pt, t, t))$ which implies that $\varphi(G(Pt, t, t)) = 0$, and hence $G(Pt, t, t) = 0$, that is, $Pt = t$.

Therefore, $Pt = Rft = t$.

Finally we prove that Qt = ght = t.

We get from (1) with x = u, y = z = t, that

$\mu\left(G(Pu, Qt, Qt)) \leq \mu(L(u, t, t)) - \varphi(L(u, t, t)\right)$ or

$\mu\left(G(t, Qt, Qt)) \leq \mu(L(u, t, t)) - \varphi(L(u, t, t)\right)$

where

$L(u, t, t) =$

$\max\{G(Rfu, ght, ght), G(Rfu, Pu, Pu), G(ght, Qt, Qt), \frac{1}{3}\{G(Rfu, Qt, Qt) + G(ght, Pu, Pu)\}$

$= \max\{G(t, Qt, Qt), G(t, t, t), G(Qt, Qt, Qt), \frac{1}{3}\{G(t, Qt, Qt) + G(Qt, t, t)\}\}$

$= \max\{G(t, Qt, Qt), 0, 0, \frac{1}{3}\{G(t, Qt, Qt) + G(Qt, t, t)\}\}$

$= \max \{G(t,Qt,Qt), \frac{2}{3}\{G(t,Qt,Qt)\}\}$, since G is symmetric.

$= G(t,Qt,Qt)$.

Hence we get,

$\mu(G(t,Qt,Qt)) \leq \mu(G(t,Qt,Qt)) - \varphi(G(t,Qt,Qt))$ which implies that $\varphi\big(G(t,Qt,Qt)\big) = 0$, and hence $G(t,Qt,Qt) = 0$, that is, $Qt = t$.

Therefore $Qt = ght = t$ and hence $Pt = Rft = Qt = ght = t$, showing that t is a common fixed point of P,Q,Rf and gh.

Uniqueness: Let s ($\neq t$) be another common fixed point of P, Q, Rf and gh. Thus, we have

$Ps = Rfs = Qs = ghs = s$.

Now from (1) with x = t, y = z = s, we obtain

$\mu(G(Pt,Qs,Qs)) \leq \mu(L(t,s,s)) - \varphi(L(t,s,s))$ or

$\mu(G(t,s,s)) \leq \mu(L(t,s,s)) - \varphi(L(t,s,s))$,

where

$L(t,s,s) = \max \{G(Rft,ghs,ghs), G(Rft,Pt,Pt), G(ghs,Qs,Qs), \frac{1}{3}\{G(Rft,Qs,Qs) + G(ghs,Pt,Pt)\}$

$= \max \{G(t,s,s), G(t,t,t), G(s,s,s), \frac{1}{3}\{G(t,s,s) + G(s,t,t)\}\}$

$= \max \{G(t,s,s), 0,0, \frac{2}{3}\{G(t,s,s)\}\}$, since G is symmetric

$= G(t,s,s)$.

Thus $\mu(G(t,s,s)) \leq \mu(G(t,s,s)) - \varphi(G(t,s,s))$ which implies that $\varphi\big(G(t,s,s)\big) = 0$, and hence $G(t,s,s) = 0$, that is, $t = s$.

Hence, P, Rf, Q, gh have a unique common fixed point in X.

We shall now prove that P, Q, R.f.g.h have a unique common fixed point.

Since (R,f),(P,R),(P,f) are commuting, we have Rt = R(Rft) = (Rf)Rt and Rt = R(Pt) = P(Rt).

Also, ft = f(Rft) = (Rf)ft and ft = f(Pt) = P(ft).This shows that Rt and ft are common fixed points of Rf and P. Hence by the uniqueness of common fixed point, we have Rt = ft = t.

Similarly, since (g,h), (Q,g), (Q,h) are commuting, we have gt = g(Qt) = (Qg)t and gt = g(ght) = gh(gt).

Also, ht = h(Qt) = Q(ht) and ht = h(ght) = gh(ht). This shows that gt and ht are common fixed points of gh and Q. Hence by the uniqueness of common fixed point, we have gt = ht = t. Therefore, Pt = Qt = Rt =ft = gt = ht =t, proving that t is a unique common fixed point of P, Q, R, f, g, h. This completes the proof.

Corollary 3.2. Let P, Q, f and h be self mappings of a symmetric G-metric space (X, G) satisfy the following conditions:

3.2.1. $\mu(G(Px,Qy,Qz)) \leq \mu(L(x,y,z)) - \varphi(L(x,y,z))$ for all $x,y,z \in X$, where μ is an altering distance function and $\varphi: [0,\infty) \to [0,\infty)$ is lower semi continuous and non-decreasing function such that $\varphi(t) = 0$ if and only if $t = 0$ and $L(x,y,z) = \max \{G(fx,hy,hz), G(fx,Px,Px), G(hy,Qy,Qz), \frac{1}{3}\{G(fx,Qy,Qz) + G(hy,Px,Px)\}$

3.2.2. (P,f) and (Q,h) satisfy $CLR_{(f,h)}$ property and

3.2.3. (P,f) and (Q,h) are weakly compatible. Then the mappings P, Q, f and h have a unique common fixed point in X.

Proof: Follows from the Theorem 3.1. by setting $R = g = I_X$ (Identity mapping).

Lemma 3.3. Let X be a set, A and B be occasionally weakly compatible mappings of X. If A and B have a unique point of coincidence w = At = Bt for some t in X, then w is the unique common fixed point of A and B.

Theorem 3.4. Let P,Q,R,f,g and h be six self maps of a symmetric G-metric space (X, G) satisfy the following conditions:

3.4.1. $\mu(G(Px, Qy, Qz)) \leq \mu(M(x, y, z)) - \varphi(M(x, y, z))$ for all $x, y, z \in X$, where μ is an altering distance function and $\varphi: [0, \infty) \to [0, \infty)$ is lower semi continuous and non-decreasing function such that $\varphi(t) = 0$ if and only if $t = 0$ and $M(x, y, z) = \max\{G(Rfx, ghy, ghz), G(Rfx, Px, Px), G(ghy, Qy, Qz), G(ghy, Px, Px)\}$ and

3.4.2. (P,Rf) and (Q,gh) are occasionally weakly compatible. Then the mappings P, Q, Rf and gh have a unique common fixed point in X. Further P, Q, R, f, g and h have a unique common fixed point in X provided the pairs of mappings (R,f), (P,R), (P,f), (g,h), (Q,g) and (Q,h) are commuting.

Proof: Since (P,Rf) and (Q,gh) are occasionally weakly compatible, we can find u and v in X such that $Pu = Rfu = u^1$ and $P(Rf)u = (Rf)Pu$ and $Qv = ghv = v^1$ and $Q(gh)v = (gh)Qv$.

We first claim that Pu = Qv.

From 3.4.1. with $x = u, y = z = v$, we get

$$\mu(G(Pu, Qv, Qv)) \leq \mu(M(u, v, v)) - \varphi(M(u, v, v)) \qquad (3)$$

where

$M((u, v. v)) =$
$\max\{G(Rfu, ghv, ghv), G(Rfu, Pu, Pu), G(ghv, Qv, Qv), G(ghv, Pu, Pu)\}$
$= \max\{G(Pu, Qv, Qv), G(Pu, Pu, Pu), G(Qv, Qv, Qv), G(Qv, Pu, Pu)\}$
$= \max\{G(Pu, Qv, Qv), 0,0, G(Qv, Pu, Pu)\}$
$= G(Pu, Qv, Qv)$, since G is symmetric.

Hence $\mu(G(Pu, Qv, Qv)) \leq \mu(G(Pu, Qv, Qv)) - \varphi(G(Pu, Qv, Qv))$ which implies that $\varphi(G(Pu, Qv, Qv)) = 0$ and so $G(Pu, Qv, Qv) = 0$, that is, $Pu = Qv$. Thus, Pu = Rfu = Qv = ghv.

Let u^1 be another point such that $Pu^1 = Rfu^1$. Then, we get, from 3.4.1., that $Pu^1 = Rfu^1 = Qv = ghv$. Hence $Pu = Pu^1$, that is, $u = u^1$, showing that P and Rf have a unique point of coincidence. Therefore by the Lemma3.3. P and Rf have a unique common fixed point, say t. Similarly it can be proved that Q and gh have a unique common fixed point, say t^1.

We now claim that $t = t^1$.

From 3.4.1. with $x = t, y = z = t^1$, we get that

$\mu(G(Pt, Qt^1, Qt^1)) \leq \mu(M(t, t^1, t^1)) - \varphi\big(M(t, t^1, t^1)\big)$ or

$$\mu(G(t, t^1, t^1)) \leq \mu(M(t, t^1, t^1)) - \varphi\big(M(t, t^1, t^1)\big) \qquad (4)$$

where

$M(t, t^1, t^1) =$
$\max\{G(Rft, ght^1, ght^1), G(Rft, Pt, Pt), G(ght^1, Qt^1, Qt^1), G(ght^1, Pt, Pt)$
$= \max\{G(t, t^1, t^1), G(t, t, t), G(t^1, t^1, t^1), G(t^1, t, t)\}$
$= \max\{G(t, t^1, t^1), 0,0,, G(t^1, t, t)\}$
$= G(t, t^1, t^1)$, since G is symmetric.

Hence, $\mu(G(t,t^1,t^1)) \leq \mu(G(t,t^1,t^1)) - \varphi\big(G(t,t^1,t^1)\big)$ which implies that $\varphi(G(t,t^1,t^1)) = 0$ and so $G(t,t^1,t^1) = 0$, that is, $t = t^1$. Thus P, Q, Rf, gh have a unique common fixed point in X. The rest of the proof is same as that of Theorem 3.1 and hence, we get that Pt = Qt = Rt =ft = gt = ht =t, proving that t is a unique common fixed point of P, Q, R, f, g, h. This completes the proof.

Corollary 3.5. Let P, Q, f and h be four self maps of a symmetric G-metric space (X,G) satisfy the following conditions:
3.5.1**.** $\mu(G(Px,Qy,Qz)) \leq \mu(M(x,y,z)) - \varphi(M(x,y,z))$ for all $x,y,z \in X$, where μ is an altering distance function and $\varphi{:}\,[0,\infty) \to [0,\infty)$ is lower semi continuous and non-decreasing function such that $\varphi(t) = 0$ if and only if $t = 0$ and $M(x,y,z) = \max\{G(fx,hy,hz), G(fx,Px,Px), G(hy,Qy,Qz), G(hy,Px,Px)\}$ and
3.5.2. (P,f) and (Q,h) are occasionally weakly compatible. Then the mappings P, Q, f and h have a unique common fixed point in X.
Proof: Follows from the Theorem 3.4. by setting R = g = I_X (Identity mapping).

4. Conclusion

In this paper, existence and uniqueness of two common fixed point theorems for six self maps in a symmetric G metric spaces have been established. In the first one common limit in the range property and weakly compatibility were utilized whereas in the second one CLR property and weakly compatibility were relaxed and occasionally weakly compatibility was utilized.

Acknowledgements. Thankful to reviewers for their valuable suggestions.

REFERENCES

1. M.A.Al-Thagafi and N Shahzad, A note on occasionally weakly compatible maps, *Int. J. Math. Anal.*, 3 (2) (2009) 55–58
2. M.S.Khan, M.Swaleh and S.Sessa, Fixed point theorems by altering distance between the points, *Bulletin of the Australian Mathematical Society*, 30 (1984) 1-9.
3. E.Karapinar, D.K.Patel, M.Imdad and D.Gopal, Some non unique fixed point theorems in symmetric spaces through $CLR_{(S,T)}$ property, *Int. J. Math. Math. Sci.*, Article ID 753965 (2013), 8 pages.
4. G.Jungck and B.E.Rhoades, Fixed point theorems for occasionally weakly compatible mappings, *Fixed Point Theory*, 7(2) (2006) 287-296.
5. Z.Mustafa and B.Sims, A new approach to generalized metric spaces, *J. Nonlinear and Convex Anal.*, 7 (2006) 289–297.
6. Z.Mustafa, H.Obiedat and F.Awawdeh, Some fixed point theorems for mapping on complete G-metric spaces, *Fixed Point Theory Appl.*, 2008, Article ID 189870, 12 pages, 2008.
7. Z.Mustafa and B.Sims, Fixed point theorems for contractive mappings in complete G-metric spaces, *Fixed Point Theory Appl.*, 2009, Article ID 917175, 2009.
8. W.Sintunavarat, P.Kumam, Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, *J. Appl. Math.*, 2011, Article ID 637958 (2011).

9. V.Nagaraju and Bathini Raju, A Common fixed point theorem using compatible mappings of type(A-1), *Annals of Pure and Applied Mathematics,* 14(2) (2017) 255-262.
10. K.Jha, M.Imdad and U.Upadhyaya, Fixed Point theorems for occasionally weakly compatible mappings in metric space, *Annals of Pure and Applied Mathematics*, 5(2) (2014) 153-157.
11. P.Malviya, V.Gupta and V.H.Badshah, Common fixed point theorem for semi compatible pairs of reciprocal continuous maps in menger spaces, *Annals of Pure and Applied Mathematics*, 11(2) (2016) 139-144.