Annals of Pure and Applied Mathematics
Vol. 22, No. 1, 2020, 65-70
ISSN: 2279-087X (P), 2279-0888(online)
Published on 25 September 2020
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v22n1a09684

Annals of Pure and Applied Mathematics

The Number of Chains of Subgroups for Certain Finite Alternating Groups

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Received 24 July 2020; accepted 21 September 2020

Abstract. In this paper, we determined the number of chains of subgroups in the subgroup lattice of certain finite alternating groups using the computational technique. It is also showed that a fuzzy subgroup is simply a chain of subgroups in the lattice of subgroups.

Keywords: Subgroups, Alternating Group, Chains of Subgroups and Fuzzy Subgroups

AMS Mathematics Subject Classification (2020): 20D05; 20D06; 20D30

1. Introduction

Throughout this paper, all groups are assumed to be finite. The lattice of subgroups of a given group G is the lattice $(L(G), \leq)$ where L(G) is the set of all subgroups of G and the partial order $\leq$ is the set inclusion. A chain of subgroups of G is called rooted (more precisely G -rooted) if it contains G, otherwise, it is called un-rooted. The study of chains of subgroups in this paper describes the set of all chains of subgroups of G that end in G. Tărnăuceanu and Bentea gave an explicit formula for the number of chains of subgroups in the lattice of a finite cyclic group by finding its generating function of one variable. The problem of counting chains of subgroups of a given group G has received attention by researchers with related to classifying fuzzy subgroups of G under a certain type of equivalence relation (see [2-6]).

2. Methodology

A chain of subgroups of G is a set of subgroups of G totally ordered by set inclusion. This paper describes the set of all chains of subgroups of G that end in G. In this way, suppose that the group G is finite and let $\mu: G \to [0,1]$ be a fuzzy subgroup of G. Put μ(G) = {α_1, α_2,…., α_r} and assume that $\alpha_1 < \alpha_2 <$….$< \alpha_r$.Then μ determines the following chain of subgroups of G which ends in G:

$$\mu G\alpha_1 \subset \mu G\alpha_2 \subset \cdots . \subset \mu G\alpha_m = G$$

Moreover, for any $x \in G$ and $i = \overline{1, r}$, we have

$$\mu(x) = \alpha_i \iff i = \max\{j | x \in \mu G\alpha_j\} \iff x \in \mu G\alpha_i \backslash \mu G\alpha_{1-r}$$

A necessary and sufficient condition for two fuzzy subgroups μ, η of G to be equivalent with respect to $\sim$ has been identified in [6] such that $\mu \sim \eta$ if and only if μ and η have the same set of level subgroups, that is they determine the same chain of subgroups. This result shows that there exists a bijection between the equivalence classes of fuzzy subgroups of G and the set of chains of subgroups of G which end in G, which is used to solve many computational problems in fuzzy group theory.

3. Main results

Let δ(G) be the number of subgroup chains of group G that terminates in G, then

$$\delta(G) = \sum_{distinct\ H \in Iso(G)} \delta(H) \times n(H) \qquad ....(\#)$$

where (i) Iso(G) is the set of representatives of isomorphism classes of subgroups of G.
(ii) n(H) denotes the size of the isomorphism class with representative H.
(iii) $\delta(H_e) = \delta(H_\alpha) = 1$, for which H_e is the trivial subgroup of G and H_α is the improper subgroup of G.

In this work (#) was used to obtain the number of chains of subgroups of G that terminates in G. We also use GAP (Group, Algorithms and Programming) to get the subgroup structures. [7]

Proposition 1. Let G be $Z_p \times Z_p$ where p is prime, then $\delta(Z_p \times Z_p)$= 2p + 4
Proof: $Z_p \times Z_p$ has the following subgroups H_e , Z_p{(p + 1) times } and $Z_p \times Z_p$ from the Isomorphism class.

Then, $\delta(Z_p \times Z_p) = \delta(H_e) + (p+1)\,\delta(Z_p) + \delta(H_\alpha) = 1 + 2(p+1) + 1 = 2p + 4$

It also follows :

(i) $\delta(Z_p) = 2$ where p is prime
(ii) $\delta(Z_{pq}) = 6$ where p and q are distinct primes
(iii) $\delta(Z_{p^2})$= 4 where p is any prime
(iv) $\delta(Z_p \times Z_p) = 2p + 4$ where p is any prime

Above result are special cases of corollary 5.2, 5.4,5.5,5.6 of [1].

3 (i). The number of chains of subgroups A_5

The alternating group A_5 is simple non-abelian group which has the following subgroups: [{e},1] , [Z_2, 15], [Z_3, 10] , [$Z_2 \times Z_2$ 15], [Z_5, 6], [S_3, 10] , [D_{10}, 6] , [A_4, 5] and [A_5, 1].

Lemma 1. Let G be Dihedral group of order 2p, where p is any prime then, δ(G)= 4 + 2p
Proof: D_{2p} has the following subgroups: [{e}, 1] , [Z_2, p] , [Z_p, 1] and [D_{2p}, 1]
$\delta(D_{2p}) = 1 + \delta(H_e) + p * \delta(Z_2) + \delta(Z_p) = 2p + 4$

Proposition 2. Let G be A_4 then, δ(G)= 24
Proof: A_4 has the following subgroups: [{e}, 1] , [Z_2, 3], [Z_3, 4] , [$Z_2 \times Z_2$, 1] and [A_4, 1]

So, we get $\delta(G) = 1 + \delta(H_e) + 3\delta(Z_2) + 4\delta(Z_3)$ $+\delta(Z_2 \times Z_2) = 24$

Lemma 2. For the symmetric groups S_3 and S_4 , we have $\delta(S_3) = 10$ and $\delta(S_4) = 232$.

Let G be A_5, then
$\delta(A_5) = 1 + \delta(H_e) + 15\delta(Z_2) + 10\delta(Z_3) + 5\delta(Z_2 \times Z_2) + 6\delta(Z_5) + 10\delta(S_3) + 6\delta(D_{10}) + 5\delta(A_4) = 408$

Theorem 1. The number of chains of subgroups of A_5 that terminates in A_5 is 408. Above result is a case of theorem 4 of [4].

3. (ii) The number of chains of subgroups for A_6

The alternating group A_6 is a simple non-abelian group of order 360 which possesses the following subgroups : [{e}, 1] ,[Z_2, 45], [Z_3, 40] , [$Z_2 \times Z_2$, 30], [Z_4, 45] , [Z_5, 360], [$Z_3 \times Z_3$, 10] , [S_3, 120] , [D_8, 45] , [D_{10}, 36] , [A_4, 30], [A_5, 12], [S_4, 30] , [($Z_3 \times Z_3$) $\ltimes Z_2$, 10], [($Z_3 \times Z_3$) $\ltimes Z_4$, 10] and [A_6, 1].

Proposition 3. Let the wreath product of the cyclic groups : $(Z_3 \times Z_3) \ltimes Z_2$ and $(Z_3 \times Z_3) \ltimes Z_4$ Then, $(Z_3 \times Z_3) \ltimes Z_2$= 158 and $(Z_3 \times Z_3) \ltimes Z_4$= 352
Proof: $(Z_3 \times Z_3) \ltimes Z_2$ has the following subgroups : [{e}, 1] ,[Z_2, 9], [$Z_3 \times Z_3$, 1], [Z_3, 4] , [S_3, 12] , [($Z_3 \times Z_3$) $\ltimes Z_2$, 1].

$\delta\big((Z_3 \times Z_3) \ltimes Z_2\big) = 1 + \delta(H_e) + 9\delta(Z_2) + \delta(Z_3 \times Z_3) + 4\delta(Z_3) + 12\delta(S_3) = 158$

$(Z_3 \times Z_3) \ltimes Z_4$ has the following subgroups : [{e}, 1] ,[Z_2, 9], [$Z_3 \times Z_3$, 1], [Z_3, 4] ,[Z_4, 9], [S_3, 12] , [($Z_3 \times Z_3$) $\ltimes Z_2$, 1] and [($Z_3 \times Z_3$) $\ltimes Z_4$, 1]

$\delta\big((Z_3 \times Z_3) \ltimes Z_4\big) = 1 + \delta(H_e) + 9\delta(Z_2) + \delta(Z_3 \times Z_3) + 4\delta(Z_3) + 9\delta(Z_4) + 12\delta(S_3) + \delta\big((Z_3 \times Z_3) \ltimes Z_2\big) = 352$

Theorem 2. The number of chains of subgroups of A_6 that terminates in A_6 is 21584
Proof: Let G be A_6 , then $\delta(A_6) = 1 + \delta(Z_1) + 45 * \delta(Z_2) + 40 * \delta(Z_3) + 30 * \delta(Z_2 \times Z_2) + 45 * \delta(Z_4) + 360 * \delta(Z_5) + 4 * \delta(S_3) + 10 * \delta(Z_3 \times Z_3) + 45 * \delta(D_8) + 30 * \delta(A_4) + 12 * \delta(A_5) + 30 * \delta(S_4)) + 36\delta(D_{10}) + 10 * \delta(Z_3 \times Z_3) \ltimes Z_2 + 10 * \delta(Z_3 \times Z_3)$ n $Z_4) = 21584$.

3. (iii) The number of chains of subgroups for A_7

The alternating group A_7 is a simple non-abelian group of degree 7, it has the following subgroups**:** [{e},1] **,** [Z_2, 105], [Z_3, 175] **,** [$Z_2 \times Z_2$, 140]**,** [Z_4, 315] **,** [Z_5, 126]**,** [$Z_3 \times Z_3$, 70] **,** [S_3, 630] **,** [D_8, 315] **,** [D_{10}, 126] **,** [D_{12}, 105], [A_4, 210] **,**[A_5, 63], [S_4, 420] **,** [S_5, 21] **,** [($Z_3 \times Z_3$) $\ltimes Z_2$, 70], [($Z_3 \times Z_3$) $\ltimes Z_4$,70], [($Z_3 \times A_4$) $\ltimes Z_2$, 35] **,** [($Z_6 \times Z_2$) $\ltimes Z_2$, 105], [$Z_3 \ltimes Z_4$, 105] **,** [$Z_3 \times A_4$, 35] **,** [$Z_5 \ltimes Z_4$, 126], [Z_6, 105] **,**[$Z_6 \times Z_2$, 35]**,** [Z_7,120]**,** [$Z_7 \ltimes Z_3$, 120] **,**[A_6, 7] **,** [PSL(3,2), 30] and [A_7, 1]

Proposition 4. Suppose that semidirect product of the cyclic groups:($Z_3 \ltimes Z_4$) and ($Z_7 \ltimes Z_3$), then $\delta(Z_3 \ltimes Z_4) = 24$ and $\delta(Z_7 \ltimes Z_3) = 18$

Proof: $Z_3 \ltimes Z_4$ which is isomorphic to A_4 has the following subgroups **:** [{e},1] **,**
[Z_2,1] **,** [Z_3,1] **,**[Z_4,3] **,** [Z_6,3] **,** and [($Z_3 \ltimes Z_4$),1]

$\delta(Z_3 \ltimes Z_4) = 1 + \delta(H_e) + \delta(Z_2) + \delta(Z_3) + 3 * \delta(Z_4) + \delta(Z_6) = 24$

$Z_7 \ltimes Z_3$ has the following subgroups **:** [{e},1] **,** [Z_3,7] **,**[Z_7,1] and [($Z_7 \ltimes Z_3$),1]

$\delta(Z_7 \ltimes Z_3) = 1 + \delta(H_e) + 7 * \delta(Z_3) + \delta(Z_7) = 18$

Lemma 3. Suppose that G be direct product of the cyclic group $Z_{2p} \times Z_2$, where p is prime number, then

$$\delta(G) = \begin{cases} 36 & \text{if } p \geq 3 \\ 24 & \text{if } p = 2 \end{cases}$$

Proof:

Case I: $p \geq 3$

$Z_{2p} \times Z_2$ has the following subgroups H_e , [Z_2,3] , Z_p , [Z_{2p},3],$Z_2 \times Z_2$ and $Z_{2p} \times Z_2$ from the Isomorphism class. .

Then, $\delta(G) = 1 + \delta(H_e) + 3 * \delta(Z_2) + \delta(Z_p) + 3 * \delta(Z_{2p}) + \delta(Z_2 \times Z_2) = 36$

Case II: p = 2 [3, Section 4**]**

($Z_4 \times Z_2$) has the following subgroups **:** [{e},1] **,** [Z_2,3]**,** [($Z_2 \times Z_2$),1] **,** [Z_4,2] and [($Z_4 \times Z_2$),1]

$\delta(G) = 1 + \delta(H_e) + 3 * \delta(Z_2) + \delta(Z_2 \times Z_2) + 2 * \delta(Z_4) = \mathbf{24}$

Lemma 4. Let G be direct product of Z_3 and A_4**,** then $\delta(Z_3 \times A_4) = 208$

($Z_3 \times A_4$) has the following subgroups **:** [{e},1] **,** [Z_2,3] **,** [$Z_2 \times Z_2$,1] **,**
[$Z_3 \times Z_3$,4] **,** [Z_3,13] **,** [Z_6,3] **,** [$Z_6 \times Z_2$,1] **,** [A_4,3] and [($Z_3 \times A_4$),1]

$\delta(G) = 1 + \delta(H_e) + 3 * \delta(Z_2) + 13 * \delta(Z_3) + 3 * \delta(Z_6) + \delta(Z_2 \times Z_2) + 4 * \delta(Z_3 \times Z_3) + \delta(Z_6 \times Z_2) + 3 * \delta(A_4) = 208$

Proposition 5. If G be the wreath product of the alternating group and the **cyclic** group $(Z_3 \times A_4) \ltimes Z_2$**,** then $\delta(G) = 5248$.

Proof: $(Z_3 \times A_4) \ltimes Z_2$ has the following subgroups **:**
[{e},1] **,** [Z_2,21] **,** [$Z_3 \times Z_3$,4] **,** [Z_3 ,13] **,** [Z_4 ,9] **,** [Z_6, 3]**,** [S_3, 12]
[$Z_2 \times Z_2$,10] **,** [$Z_3 \times Z_3$, 4]**,** [$Z_6 \times Z_2$,1] **,** [$Z_3 \times A_4$,1] **,** [$Z_3 \ltimes Z_4$,3] **,** [($Z_3 \times Z_3) \ltimes Z_2$,4]
[A_4,3] **,** [S_3,42] **,** [S_4,9] **,** [D_8,9] **,** [D_{12},3] **,**[($Z_6 \times Z_2) \ltimes Z_2$,3] and
[($Z_3 \times A_4) \ltimes Z_2$,1].

$\delta(G) = 1 + \delta(H_e) + 21 * \delta(Z_2) + 13 * \delta(Z_3) + 9(\delta(Z_4)) + 3 * \delta(Z_6)$
$+ 10 * \delta(Z_2 \times Z_2) + 1(\delta(Z_6 \times Z_2)) + 4*\delta(Z_3 \times Z_3) + \delta(Z_3 \times A_4) + 3*\delta(Z_3 \ltimes Z_4)$
$+3*\delta(A_4) + 3* \delta(D_{12} + 9*\delta(D_8) + 42*\delta(S_3) + 9*\delta(S_4) + 4*\delta(Z_3 \times Z_3) \ltimes Z_2$
$+3*\delta(Z_6 \times Z_2) \ltimes Z_2 = 5248$

Proposition 6. Suppose that G be the wreath product of the cyclic group**:** $((Z_6 \times Z_2) \ltimes Z_2$**,** then $\delta(G) = 328$.

Proof: $(Z_6 \times Z_2) \ltimes Z_2$**)** has the following subgroups **:** $[\{e\},1]$ **,** $[Z_2,9]$ **,** $[Z_3 \times Z_3,1]$ **,** $[Z_3, 1]$ **,** $[Z_4, 3]$**,** $[Z_6, 3]$**,** $[S_3, 2]$ **,** $[Z_2 \times Z_2,4]$**,** $[Z_3 \ltimes Z_4,1]$ **,** $[Z_6 \times Z_2,1]$ **,** $[D_8, 3]$,**[**$D_{12},1]$ and $[(Z_6 \times Z_2) \ltimes Z_2),1]$

$\delta(G) = 1 + \delta(H_e) + 9 * \delta(Z_2) + \delta(Z_3) + 3 * \delta(Z_4) + 3 * \delta(Z_6) + 4 * \delta(Z_2 \times Z_2) + \delta(Z_6 \times Z_2) + 4 * \delta + 2 * \delta(S_3) + \delta(D_{12}) + 3 * \delta(D_8) = 328$

Lemma 5. If G be the projective special linear group PSL(3,2), then $\delta(PSL(3,2)) = 4992$**.**

Proof: $PSL(3,2)$ is simple non-abelian group of order 168, it has the following subgroups **:** $[\{e\},1]$ **,** $[Z_2,21]$ **,** $[Z_3,28]$ **,** $[Z_4,21]$ **,** $[Z_7,8]$**,** $[S_3,28]$ $[Z_2 \times Z_2,14]$ **,** $[Z_7 \ltimes Z_3,8]$ **,** $[A_4,14]$ **,** $[S_3,28]$ **,** $[S_4,14]$ **,** $[D_8,21]$ and $[PSL(3,2),1]$

$\delta(G) = 1+\delta(H_e)+21*\delta(Z_2)+28*\delta(Z_3)+21*\delta(Z_4)+8*\delta(Z_7)+14*\delta(Z_2 \times Z_2) + 8 * \delta(Z_7 \ltimes Z_3) + 14 * \delta(A_4) + 21 * \delta(D_8) + 28 * \delta(S_3) + 14 * \delta(S_4) = 4992$

Theorem 3. The number of chains of subgroups of A_7 that terminates in A_7 is 811632

Proof: Let G be A_7**,** then

$\delta(G) = 1+\delta(H_e)+ 105*\delta(Z_2)+ 175*\delta(Z_3)+ 140*\delta(Z_2 \times Z_2)+315*\delta(Z_4)+126* \delta(Z_5)+ 105*\delta(Z_6)+120*\delta(Z_7)+ 70*\delta(Z_3 \times Z_3)+ 35*\delta(Z_6 \times Z_2)+ 105*\delta(Z_3 \ltimes Z_4)+35* \delta(Z_3 \times A_4)+ 126*\delta(Z_5 \ltimes Z_4)+ 210*\delta(A_4)+ 65*\delta(A_5)+7*\delta(A_6)+ 315*\delta(D_8)+126* (\delta(D_{10})+ 105*(\delta(D_{12})+ 630*\delta(S_3)+ 420*\delta(S_4))+21*\delta(S_5) +30*\delta(PSL(3,2))+ 35*\delta(Z_3 \times A_4) \ltimes Z_2 +70*\delta(Z_3 \times Z_3) \ltimes Z_2+70*\delta(Z_3 \times Z_3) \ltimes Z_4 +105*\delta(Z_6 \times Z_2) \ltimes Z_2$
$= 811632$.

4. Conclusion

In this paper, we treated to counting the number of chains of subgroups for some specific alternating groups which is also the exact number of distinct fuzzy subgroups by the natural equivalence relation $\sim$ as used in [4]. This will surely constitute the subject of further research on the classification of the fuzzy subgroups of finite alternating groups.

REFERENCES

1. A.Sehgal, S.Sehgal, P.K.Sharma and M.Jakhar, Fuzzy subgroups of a finite abelian group, *Advances in Fuzzy Sets and Systems*, 21 (2016) 291-302.
2. J.-M.Oh, The number of chains of subgroups of a finite cycle group, *European Journal of Combinatorics*, 33(2) (2012) 259–266.
3. L.K.Ardekania and B.Davvaz, Classifying fuzzy (normal) subgroups of the group $D_{2p} \times Z_q$ and finite groups of order $n \leq 20$, *Journal of Intelligent Fuzzy Systems*, 33 (2017) 3615–3627.

4. M.Ogiugo and M.Eni Oluwafe, Classifying a class of the fuzzy subgroups of the alternating group A_n, *Imhotep Mathematical Proceedings* , 4 (2017) 27- 33
5. M.Tărnăuceanu and L.Bentea, On the number of fuzzy subgroups of finite abelian groups, *Fuzzy Sets and Systems,* 159 (2008) 1084–1096.
6. A.C.Volf, Counting fuzzy subgroups and chains of subgroups. *Fuzzy Systems & Artificial Intelligence*, 10 (2004) 191–200.
7. GAP–Groups, Algorithms and Programming, Version 4.8.7; (https://www.gap-system.org)