Brief Note

Verification of a Conjecture Proposed by N. Burshtein on a Particular Diophantine Equation

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Abstract. In [1] among other equations, the author considered the equation \(p^x + (p + 1)^y + (p + 2)^z = M^2\) when \(p = 4N + 3\) is prime, \(x = 1, y = z = 2\) and \(M\) is a positive integer. For all values \(0 \leq N \leq 50\), he established that the equation has exactly one solution when \(N = 2\), namely when \(p = 11\). In [1 – Conjecture 1] he stated that the equation has no solutions for all values \(N > 50\). In this note we verify that Conjecture 1 is indeed true for all values \(N > 50\).

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation \(p^x + q^y = z^2\) has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds.

In [1], we extended the above equation, and considered equations of the form \(p^x + (p + 1)^y + (p + 2)^z = M^2\) for all primes \(p \geq 2\) and integers \(x, y, z\) satisfying \(1 \leq x, y, z \leq 2\). The value \(M\) is a positive integer. All the possibilities for infinitely many solutions, no solution cases and unique solutions have been determined, except for the equation \(p + (p + 1)^2 + (p + 2)^2 = M^2\) when \(p\) is of the form \(4N + 3\). In this case, it was established that \(p = 11\) is the only solution when \(3 \leq p \leq 199\). We have conjectured [1 – Conjecture 1] that for all primes \(p > 199\), the equation has no solutions. In this note, we provide a formal proof as to the validity of our conjecture in [1] implying now that the solution with \(p = 11\) is unique.

2. All the solutions of \(p + (p + 1)^2 + (p + 2)^2 = M^2\) when \(p = 4N + 3\)

In the following theorem we will show that the equation has a unique solution.
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**Theorem 2.1.** Suppose that \( p = 4N + 3 \) (\( N \geq 0 \)) is prime. Then the equation \( p + (p + 1)^2 + (p + 2)^2 = M^2 \) has a unique solution when \( p = 11 \) (\( N = 2 \)).

**Proof:** The left side of the equation yields
\[
p + (p + 1)^2 + (p + 2)^2 = 2p^2 + 7p + 5 = (p + 1)(2p + 5) = (p + 1)(p + 1) + 3.
\]
(1)

If \( (p + 1)(2(p + 1) + 3) = M^2 \) has a solution for some value \( p \), then the two factors \( (p + 1) \), \( (2(p + 1) + 3) \) in (1) must satisfy simultaneously the two conditions in each of the following cases, namely:

(a) \( p + 1 = A^2, \quad 2(p + 1) + 3 = B^2 \).
(b) \( p + 1 \neq A^2, \quad 2(p + 1) + 3 \neq B^2 \).

Suppose (a): \( p + 1 = A^2, \quad 2(p + 1) + 3 = B^2 \).
The equality \( p + 1 = A^2 \) implies that \( p = A^2 - 1 = A^2 - 1^2 = (A - 1)(A + 1). \) When \( A = 2, \) then \( p = 3. \) But \( 2(3 + 1) + 3 = 11 \neq B^2 \). Thus \( A \neq 2. \) For all values \( A > 2, \) the prime \( p = (A - 1)(A + 1) \) is a product of two distinct factors which is impossible. The two conditions in (a) are not satisfied simultaneously.

Hence case (a) does not exist.

Suppose (b): \( p + 1 \neq A^2, \quad 2(p + 1) + 3 \neq B^2 \).
We have two cases, namely \( \gcd (p + 1, 2(p + 1) + 3) = 1, \) \( \gcd (p + 1, 2(p + 1) + 3) = 3. \)

If \( \gcd (p + 1, 2(p + 1) + 3) = 1, \) and \( (p + 1)(2(p + 1) + 3) = M^2, \) it then follows that \( p + 1 = A^2 \) and \( 2(p + 1) + 3 = B^2 \) must exist simultaneously. But this contradicts our supposition, and hence \( \gcd (p + 1, 2(p + 1) + 3) \neq 1. \)

If \( \gcd (p + 1, 2(p + 1) + 3) = 3, \) denote \( p + 1 = 3K, \) and \( 2(p + 1) + 3 = 2 \cdot 3K + 3 = 3(2K + 1) \) where \( \gcd (K, 2K + 1) = 1. \) If \( (p + 1)(2(p + 1) + 3) = 3 \cdot 2(2K + 1) = 3 \cdot (2K + 1)M^2, \) it now follows that the two conditions \( K = H^2 \) and \( 2K + 1 = 2H^2 + 1 = L^2 \) exist simultaneously. In order to achieve the smallest possible difference \( L^2 - 2H^2 = 1, \) set \( H \) as the largest possible value \( H = L - 1. \) We then obtain
\[
L^2 - 2H^2 = L^2 - 2(L - 1)^2 = -L^2 + 4L - 2 = L(4 - L) - 2.
\]
(2)
Since for all values \( L \geq 4, \) it follows from (2) that \( L(4 - L) - 2 < 0, \) therefore \( L \) may assume only the two values \( L = 2, 3. \) When \( L = 2, \) then in (2) \( L^2 - 2H^2 = 2 > 1, \) and hence \( H = 2. \) This in turn implies that \( K = H^2 = 4, \) \( p + 1 = 3K = 12 \) and \( p = 11 \) for which \( M = 18. \) When \( p = 11, \) it follows that the two conditions in which \( p + 1 = 12 \neq A^2, \) and \( 2(p + 1) + 3 = 27 \neq B^2 \) are indeed satisfied simultaneously.

The equation \( p + (p + 1)^2 + (p + 2)^2 = M^2 \) has a unique solution in which \( p = 11 \) and \( M = 18. \)

This concludes the proof of Theorem 2.1. \( \Box \)

**Final remark.** In [1] we have shown that when \( 3 \leq p \leq 199, \) the equation \( p + (p + 1)^2 + (p + 2)^2 = M^2 \) has exactly one solution with \( p = 11. \) Theorem 2.1 establishes that
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Conjecture 1 in [1] which stated that for all \( p > 199 \) the equation has no solutions is indeed true now, and the solution with \( p = 11 \) is therefore unique.

REFERENCES

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