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All the Solutions of the Diophantine Equations $13^x - 5^y = z^2$, $19^x - 5^y = z^2$ in Positive Integers x, y, z

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Dedicated to Ali Burshtein

Abstract. In this article, the author has investigated the equations $13^x - 5^y = z^2$ and $19^x - 5^y = z^2$ with positive integers x, y, z. It was established that $13^x - 5^y = z^2$ has a unique solution, whereas $19^x - 5^y = z^2$ has no solutions.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds. Among them are for example [2, 4, 6, 8].

In this article, we consider the two equations

$$13^x - 5^y = z^2$$
$$19^x - 5^y = z^2$$

in which x, y, z are positive integers. Although resemblance exists between the two equations, they nevertheless differ in one point namely, 13 is a prime of the form $4N + 1$, whereas 19 is a prime of the form $4N + 3$. It is our interest to find all the solutions for these two equations. This is done in Sections 2 and 3, where all theorems are self-contained.

2. All the solutions of the equation $13^x - 5^y = z^2$

In this section we will show that the equation $13^x - 5^y = z^2$ in positive integers x, y, z has exactly one solution.

Theorem 2.1. The equation $13^x - 5^y = z^2$ in positive integers x, y, z has a unique solution.

Proof: We shall assume two cases, namely when x is odd and when x is even.
Suppose that x is odd.

For all values $y \geq 1$, the power 5^y ends in the digit 5. When x is odd, then 13^x ends in the digit 3 or in the digit 7. Therefore, the difference $13^x - 5^y$ respectively ends in the digit 8 or in the digit 2. Since for all values x, y, the above difference is even, it follows that if $13^x - 5^y = z^2$, then z^2 is even. An even square cannot end in the digit 8 or end in the digit 2. Therefore, when x is odd $13^x - 5^y \neq z^2$.

Suppose that x is even.

Let $x = 2n$ where $n \geq 1$ is an integer. If $13^{2n} - 5^y = z^2$ has a solution for some value z, then $5^y = 13^{2n} - z^2 = (13^n)^2 - z^2$ or

$$5^y = (13^n - z)(13^n + z). \tag{1}$$

Denote in (1)

$$13^n - z = 5^A, \qquad 13^n + z = 5^B, \qquad A < B, \qquad A + B = y,$$

where A, B are non-negative integers. The sum $5^A + 5^B$ yields

$$2\cdot 13^n = 5^A(5^{B-A} + 1). \tag{2}$$

If $A > 0$, then $5 \nmid 2\cdot 13^n$. Therefore $A \not> 0$. When $A = 0$, then $B = y$. Hence (2) results in

$$2\cdot 13^n = 5^y + 1. \tag{3}$$

If (3) holds, it then follows that $n = 4m + 1$ where $m \geq 0$ is an integer. The values $n = 1$ $(m = 0)$ and $y = 2$ yield a solution of (3), namely $2\cdot 13^1 = 5^2 + 1$. Thus, the equation $13^x - 5^y = z^2$ has the solution

$$13^2 - 5^2 = 12^2. \tag{4}$$

The uniqueness of (4) is now determined as follows. Let $n \geq 2$. Writing (3) as $2\cdot 13^n = 5^y + 1^y$, then for all odd values $y \geq 3$, we obtain the identity

$$5^y + 1^y = (5 + 1)(5^{y-1} - 5^{y-2} + 5^{y-3} - \cdots - 5^1 + 1). \tag{5}$$

In (5), the factor $(5 + 1)$ is a multiple of 3, but in (3) $3 \nmid 2\cdot 13^n$. Therefore, for all odd values $y \geq 3$, equation (3) has no solutions.

Suppose that y is even, and let $y = 2a$ where $a \geq 2$ is an integer. Then (3) implies $2\cdot 13^n - 5^y = 1$ or

$$2\cdot 13^{4m+1} - 5^{2a} = 1. \tag{6}$$

Certainly, the value 1 in (6) may be achieved only if a is the largest possible value for which (6) holds. We shall now examine the first three possibilities of (6) when $m = 1, 2, 3$, with their respective largest possible values a.

Let $m = 1$. Then $a = 4$, and we have

$$2\cdot 13^5 - 5^8 = 742586 - 390625 = 351961 \neq 1.$$

Let $m = 2$. Then $a = 7$, and we have

$$2\cdot 13^9 - 5^{14} = 21208998746 - 6103515625 = 15105483121 \neq 1.$$

Let $m = 3$. Then $a = 10$, and we have

$$2\cdot 13^{13} - 5^{20} = 605750213184506 - 95367431640625 = 510382781543881 \neq 1.$$

Decisively and without any doubt, it follows from the above three cases that when m and a are increasing, so does the difference of the left side of (6). The value 1 in (6) is never attained, and y is not even. Equation (6) has no solutions.

Equality (4) is therefore unique, and the equation $13^x - 5^y = z^2$ has exactly one solution as asserted.

The proof of Theorem 2.1 is complete. □

3. All the solutions of the equation $19^x - 5^y = z^2$

In this section we consider the equation $19^x - 5^y = z^2$ in positive integers x, y, z. In Theorem 3.1 we establish that the equation has no solutions.

Theorem 3.1. The equation $19^x - 5^y = z^2$ in positive integers x, y, z has no solutions.

Proof: We shall consider two cases, namely $x = 2n$ and $x = 2n + 1$, where $n \geq 0$ is an integer.

Suppose that $x = 2n$.

We shall assume that $19^{2n} - 5^y = z^2$ has a solution, and reach a contradiction.

By our assumption, we have $19^{2n} - 5^y = z^2$ implying that $5^y = 19^{2n} - z^2 = (19^n)^2 - z^2$ or

$$5^y = (19^n - z)(19^n + z). \tag{7}$$

Denote in (7)

$$19^n - z = 5^A, \qquad 19^n + z = 5^B, \qquad A < B, \qquad A + B = y,$$

where A, B are non-negative integers. Then $5^A + 5^B$ yields $2\cdot 19^n = 5^A + 5^B$ or

$$2\cdot 19^n = 5^A(5^{B-A} + 1). \tag{8}$$

If $A > 0$, then $5 \nmid 2\cdot 19^n$, and therefore $A \not> 0$. When $A = 0$, then $B = y$ and (8) results in

$$2\cdot 19^n = 5^y + 1. \tag{9}$$

For all values $y \geq 1$, the power 5^y ends in the digit 5, and hence $5^y + 1$ ends in the digit 6. For all values $n \geq 1$, the power 19^n ends either in the digit 9 or in the digit 1. Thus, $2\cdot 19^n$ respectively ends either in the digit 8 or in the digit 2. Both sides of (9) now end in two distinct digits such as 8 and 6 or 2 and 6. The equality in (9) is therefore impossible. This contradiction implies that (9) has no solutions, and hence $x \neq 2n$.

Suppose that $x = 2n + 1$.

Since $5 = 4N + 1$ $(N = 1)$, then for all values $y \geq 1$ the power 5^y is of the form $4U + 1$ where U is an integer. The prime $19 = 4M + 3$ $(M = 4)$, and for all values $n \geq 0$ the power 19^{2n+1} is of the form $4V + 3$ where V is an integer. If $19^{2n+1} - 5^y = z^2$ has a solution, then z^2 is even, and $z^2 = 4T^2$. The difference $19^{2n+1} - 5^y$ has the form $(4V + 3) - (4U + 1) = 4(V - U) + 2 \neq 4T^2 = z^2$. This implies that $x \neq 2n + 1$, and $19^{2n+1} - 5^y \neq z^2$.

Since no value x exists which satisfies the equation $19^x - 5^y = z^2$, it follows that the equation has no solutions as asserted.

This concludes the proof of Theorem 3.1. □

Corollary 3.1. Consider the equation $p^x - q^y = z^2$. Suppose that $p = 4M + 3$ $(M \geq 0)$ is prime, and $x = 2n + 1$. Furthermore, suppose that $q = 4N + 1$ $(N \geq 1)$ is prime, and $y \geq 1$. Then the equation $p^x - q^y = z^2$ has no solutions.

Proof: When 19 is replaced by a prime $p = 4M + 3$, then $(4M + 3)^{2n+1}$ is of the form $4V + 3$ for all M, n. When 5 is replaced by a prime $q = 4N + 1$, then $(4N + 1)^y$ is of the form $4U + 1$ for all $N > 0$, $y \geq 1$. The result which follows from the proof of the second part $(x = 2n + 1)$ of Theorem 3.1 is also valid now, and $(4M + 3)^{2n+1} - (4N + 1)^y \neq z^2$. Moreover, the result holds when one of p, q or both are composites of the same forms.

Final remark. It has been shown in this article that the equation $13^x - 5^y = z^2$ has a unique solution, whereas the equation $19^x - 5^y = z^2$ has no solutions. The results were achieved mainly by our new elementary technique which uses the last digits of the powers involved.
We note that to the best of our knowledge, other authors have not considered equations of the kind $p^x - q^y = z^2$. It is therefore obvious, that no references concerning such equations can be provided.

REFERENCES

1. N. Burshtein, On the diophantine equations $p^4 + q^3 = z^2$ and $p^4 - q^3 = z^2$ when p, q are distinct odd primes, *Annals of Pure and Applied Mathematics*, 21 (1) (2020) 15-17.
2. N. Burshtein, A short note on solutions of the diophantine equations $6^x + 11^y = z^2$ and $6^x - 11^y = z^2$ in positive integers x, y, z, *Annals of Pure and Applied Mathematics*, 20 (2) (2019) 55-56.
3. N. Burshtein, All the solutions of the diophantine equations $p^x + p^y = z^2$ and $p^x - p^y = z^2$ when $p \geq 2$ is prime, *Annals of Pure and Applied Mathematics*, 19 (2) (2019) 111-119.
4. N. Burshtein, On solutions of the diophantine equations $p^4 + q^4 = z^2$ and $p^4 - q^4 = z^2$ when p and q are primes, *Annals of Pure and Applied Mathematics*, 19 (1) (2019) 1-5.
5. N. Burshtein, On solutions of the diophantine equations $p^3 + q^3 = z^2$ and $p^3 - q^3 = z^2$ when p, q are primes, *Annals of Pure and Applied Mathematics*, 18 (1) (2018) 51-57.
6. S. Kumar, S. Gupta and H. Kishan, On the non-linear diophantine equation $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$, *Annals of Pure and Applied Mathematics*, 18 (1) (2018) 91-94.
7. B. Poonen, Some diophantine equations of the form $x^n + y^n = z^m$, *Acta. Arith.*, 86 (1998) 193-205.
8. B. Sroysang, On the diophantine equation $5^x + 7^y = z^2$, *Int. J. Pure Appl. Math.*, 89 (2013) 115-118.
9. B. Sroysang, On the diophantine equation $3^x + 17^y = z^2$, *Int. J. Pure Appl. Math.*, 89 (2013) 111-114.