Annals of Pure and Applied Mathematics
Vol. 23, No. 1, 2021, 1-6
ISSN: 2279-087X (P), 2279-0888(online)
Published on 22 January 2021
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v23n1a01808

Annals of
Pure and Applied Mathematics

Hopf Bifurcation of a Rosenzweig-Macarthur Hyperbolic Tangent-Type Predator-Prey Model

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Received 16 December 2020; accepted 19 January 2021

Abstract. In this paper, the Rosenzweig-MacArthur predator-prey model with the hyperbolic tangent functional response is investigated. We choose capturing efficiency of predator as the bifurcation parameter and mainly discuss the existence, direction and stability of Hopf bifurcation by Poincaré-Andronov-Hopf bifurcation theorem.

Keywords: Rosenzweig-MacArthur predator-prey model, Hyperbolic tangent functional response, Hopf bifurcation.

AMS Mathematics Subject Classification (2010): 92D40

1. Introduction

In recent years, 'paradox of enrichment' is a hot topic in both mathematical biology and population dynamics. More and more people give the corresponding explanations about this phenomenon. At present, some scholors work on using curve fitting method and graphical representation way to predict the population dynamics of the model. It can be referred to [1,2,3]. Their research direction tends to mathematical analysis and numerical simulation, rather than biological phenomena or experimental results. It means that the mathematical form of the model may play an important role.

Back in 2005, Fussmann and Blasius [4] considered a classical Rosenzweig-MacArthur predator-prey model as follows

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = g(u) - \Phi(u)v, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = (\Phi(u) - m)v, \end{cases} \tag{1}$$

where $u(t)$ and $v(t)$ stand for the prey population and predator population, respectively. r is prey intrinsic growth rate, K is carrying capacity, m is per capita mortality rate. Predator and prey population grow logistically at the rate $g(u)$ and the nonlinear functional response $\Phi(u)$.

Fussmann and Blasius [4] found a different result is that 'paradox of enrichment' phenomenon may depend on the mathematical form of functional response function.

As is known to all, model (1) has three different response functions: Holling type Ⅱ [5], Ivlev [6], and hyperbolic tangent [7] as follows

$$\Phi_H(u) = \frac{a_H(u)}{1+b_H(u)},$$
$$\Phi_I(u) = a_I\left(1-e^{-b_I(u)}\right),$$
$$\Phi_T(u) = a_T\tanh\left(b_T(u)\right).$$

There is little literature that studies the R-M model with above three different response functions. Naturally, no one comes to a valuable conclusion. It was not until 2018 that Seo and Wolkowicz [8] concentrated on the general Rosenzweig-MacArthur predator-prey model with three different response functions. The conclusion is that in the case of hyperbolic tangent functional response, system (1) exhibits richer dynamics.

Hence, inspired by Seo and Wolkowicz [8], we discuss the following modified model in this new direction of research

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = ru\left(1-\dfrac{u}{K}\right) - a\tanh(cu)v, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \left(a\tanh(cu)-m-h\right)v, \end{cases} \tag{2}$$

where a is the conversion rate, c implies the efficiency of the predator for capturing prey. h is the harvesting effort. And other parameters are introduced in (1).

Obviously, model (2) has a trivial equilibrium $E_0=(0,0)$, a semi-trivial equilibrium $E_K=(K,0)$, and a unique positive equilibrium $E^*=\left(u^*,v^*\right)$ if and only if

$$(\mathrm{H}_0) \quad \operatorname{arctanh}\left(\frac{m+h}{a}\right) < cK, \quad a > m+h.$$

where

$$u^* = \frac{1}{c}\operatorname{arctanh}\left(\frac{m+h}{a}\right), \quad v^* = \frac{r}{c(m+h)}\operatorname{arctanh}\left(\frac{m+h}{a}\right)\left[1-\frac{1}{cK}\operatorname{arctanh}\left(\frac{m+h}{a}\right)\right].$$

2. Hopf bifurcation

We choose capturing efficiency c of predator as the bifurcation parameter to seek the condition for Hopf bifurcation of model (2) occurring at E^*. We verify that c_0 is the unique positive root of $\Theta=0$ in the following.

Set

$$y(b) := \left(a^2-b^2\right)\operatorname{arctanh}\left(\frac{b}{a}\right) - ab, \quad b = m+h > 0.$$

Notice that

$$y(0)=0, \quad y'(b) = -2b\operatorname{arctanh}\left(\frac{b}{a}\right) < 0.$$

Hence, $y(b)<0, \ c_0>0.$

Suppose that $\lambda(c)=\alpha(c)\pm i\beta(c)$ be a pair of complex roots of $P(\lambda)=0$ when c is near c_0, then

$$\alpha(c) = \frac{a_{11}}{2}, \quad \beta(c) = \sqrt{4(m+h)a_{21}-a_{11}^2}.$$

Simple computations show that

$$\alpha'(c_0)=\frac{rK\left[\left(a^2-(m+h)^2\right)\operatorname{arctanh}\left(\frac{m+h}{a}\right)-a(m+h)\right]^2}{2a(m+h)\operatorname{arctanh}\left(\frac{m+h}{a}\right)\left[2a(m+h)-\left(a^2-(m+h)^2\right)\operatorname{arctanh}\left(\frac{m+h}{a}\right)\right]}>0.$$

Thus, system (2) undergoes a Hopf bifurcation at E^* as c passes through c_0.

In the following, we perform a further analysis for the normal form to study the detailed property of Hopf bifurcation. We make the transformation $\tilde{u}=u-u^*$, $\tilde{v}=v-v^*$. For convenience, we still denote $\tilde{u}$ and $\tilde{v}$ by u and v, respectively. Thus, system (2) is transformed by

$$\begin{cases}\frac{\mathrm{d}u}{\mathrm{d}t}=r\left(u+u^*\right)\left(1-\frac{u+u^*}{K}\right)-a\tanh\left(c\left(u+u^*\right)\right)\left(v+v^*\right),\\ \frac{\mathrm{d}v}{\mathrm{d}t}=\left(a\tanh\left(c\left(u+u^*\right)\right)-m-h\right)\left(v+v^*\right).\end{cases}\tag{3}$$

System (3) can be written as

$$\begin{pmatrix}\frac{\mathrm{d}u}{\mathrm{d}t}\\ \frac{\mathrm{d}v}{\mathrm{d}t}\end{pmatrix}=J\begin{pmatrix}u\\ v\end{pmatrix}+\begin{pmatrix}f(u,v,c)\\ g(u,v,c)\end{pmatrix},\tag{4}$$

where

$$f(u,v,c)=a_1u^2+a_1uv+a_3v^2+a_4u^3+a_5u^2v+a_6uv^2+\cdots,$$
$$g(u,v,c)=b_1u^2+b_2uv+b_3v^2+b_4u^3+b_5u^2v+b_6uv^2+\cdots,$$

and

$$a_1=\frac{(m+h)Kc^2\left(a^2-(m+h)^2\right)v^*-ra^2}{Ka^2},\qquad a_2=-\frac{c\left(a^2-(m+h)^2\right)}{a},\qquad a_3=0,$$

$$a_4=\frac{c^3\left(a^2-(m+h)^2\right)\left(a^2-3(m+h)^2\right)v^*}{3a^3},\qquad a_5=\frac{mc^2\left(a^2-(m+h)^2\right)}{a^2},\qquad a_6=0,$$

$$b_1=-\frac{(m+h)c^2\left(a^2-(m+h)^2\right)v^*}{a^2},\qquad b_2=\frac{c\left(a^2-(m+h)^2\right)}{a},\qquad b_3=0,$$

$$b_4=-\frac{c^3\left(a^2-(m+h)^2\right)\left(a^2-3(m+h)^2\right)v^*}{3a^3},\qquad b_5=-\frac{(m+h)c^2\left(a^2-(m+h)^2\right)}{a^2},\qquad b_6=0.$$

Set the matrix

$$\mathrm{P}:=\begin{pmatrix}B&1\\ A&0\end{pmatrix},$$

where $A=-\frac{a_{21}}{\beta}$, $B=\frac{a_{22}-a_{11}}{2\beta}$. Thus,

$$\mathrm{P}^{-1}J\mathrm{P}=\Phi(c):=\begin{pmatrix}\alpha(c) & -\beta(c)\\ \beta(c) & \alpha(c)\end{pmatrix}.$$

Assume that

$$A_0:=A\big|_{c=c_0},\quad B_0:=B\big|_{c=c_0},\quad \beta_0:=\beta(c_0).$$

By the transformation $(u,v)^T=\mathrm{P}(x,y)^T$, model (4) is written as follows

$$\begin{pmatrix}\dfrac{\mathrm{d}x}{\mathrm{d}t}\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}\end{pmatrix}=\Phi(c)\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}f^1(u,v,c)\\ g^1(u,v,c)\end{pmatrix}, \tag{5}$$

where

$$\begin{aligned}
f^1(u,v,c)&=\frac{1}{A}g(Bx+y,Ax,c)\\
&=\left(\frac{B^2}{A}b_1+Bb_2\right)x^2+\left(\frac{2B}{A}b_1+b_2\right)xy+\frac{b_1}{A}y^2+\left(\frac{B^3}{A}b_4+B^2b_5\right)x^3\\
&\quad+\left(\frac{3B^2}{A}b_4+2Bb_5\right)x^2y+\left(\frac{3B}{A}b_4+b_5\right)xy^2+\frac{b_4}{A}y^3+\cdots,\\
g^1(u,v,c)&=f(Bx+y,Ax,c)-\frac{B}{A}g(Bx+y,Ax,c)\\
&=\left(B^2a_1+ABa_2-\frac{B^3}{A}b_1-B^2b_2\right)x^2+\left(2Ba_1+Aa_2-\frac{2B^2}{A}b_1-Bb_2\right)xy\\
&\quad+\left(a_1-\frac{B}{A}b_1\right)y^2+\left(B^3a_4+B^2Aa_5-\frac{B^4}{A}b_4-B^3b_5\right)x^3\\
&\quad+\left(3B^2a_4+2ABa_5-\frac{3B^3}{A}b_4-2B^2b_5\right)x^2y\\
&\quad+\left(3Ba_4+Aa_5-\frac{3B^2}{A}b_4-Bb_5\right)xy^2+\left(a_4-\frac{B}{A}b_4\right)y^3+\cdots.
\end{aligned}$$

The polar coordinate form of (5) is as follows

$$\begin{aligned}\dot\rho&=\alpha(c)\rho+a(c)\rho^3+\cdots,\\ \dot\theta&=\beta(c)\rho+d(c)\rho^2+\cdots,\end{aligned}\tag{6}$$

then the Taylor expansion of (6) at $c=c_0$ is

$$\begin{aligned}\dot\rho&=\alpha'(c_0)(c-c_0)\rho+a(c_0)\rho^3+o\Big((c-c_0)^2\rho,(c-c_0)\rho^3,\rho^5\Big),\\ \dot\theta&=\beta(c_0)+\beta'(c_0)(c-c_0)+d(c_0)\rho^2+o\Big((c-c_0)^2,(c-c_0)\rho^2,\rho^4\Big).\end{aligned}$$

In order to understand the stability of Hopf bifurcation periodic solution, we need to calculate the sign of $a(c_0)$, the formulation yields

$$a(c_0):=\frac{1}{16}\left(f^1_{xxx}+f^1_{xyy}+g^1_{xxy}+g^1_{yyy}\right)+\frac{1}{16\beta_0}\left[f^1_{xy}\left(f^1_{xx}+f^1_{yy}\right)-g^1_{xy}\left(g^1_{xx}+g^1_{yy}\right)-f^1_{xx}g^1_{xx}+f^1_{yy}g^1_{yy}\right],$$

where all partial derivative are evaluated at the bifurcation point $(x,y,c)=(x,y,c_0)$ as follows

$$f^1_{xxx}(0,0,c_0)=6\left(\frac{B_0^3}{A_0}b_4+B_0^2b_5\right),\qquad f^1_{xyy}(0,0,c_0)=2\left(\frac{3B_0}{A_0}b_4+b_5\right),$$

$$g^1_{xxy}(0,0,c_0)=2\left(3B_0^2a_4+2A_0B_0a_5-\frac{3B_0^3}{A_0}b_4-2B_0^2b_5\right),\qquad g^1_{yyy}(0,0,c_0)=6\left(a_4-\frac{B_0}{A_0}b_4\right),$$

$$f^1_{xx}(0,0,c_0)=2\left(\frac{B_0^2}{A_0}b_1+B_0b_2\right),\qquad f^1_{xy}(0,0,c_0)=\frac{2B_0}{A_0}b_1+b_2,$$

$$f^1_{yy}(0,0,c_0)=\frac{2}{A_0}b_1,\qquad g^1_{xx}(0,0,c_0)=2\left(B_0^2a_1+A_0B_0a_2-\frac{B_0^3}{A_0}b_1-B_0^2b_2\right),$$

$$g^1_{xy}(0,0,c_0)=2B_0a_1+A_0a_2-\frac{2B_0^2}{A_0}b_1-B_0b_2,\qquad g^1_{yy}(0,0,c_0)=2\left(a_1-\frac{B_0}{A_0}b_1\right).$$

Note that $a_2=-b_2, a_4=-b_4, a_5=-b_5$ and $b_1=v^*b_5$. Hence, we have

$$\begin{aligned}a(c_0)=&-\frac{B_0^3+B_0}{4\beta_0}a_1^2+\frac{B_0^3+3B_0^2A_0+B_0+A_0}{8\beta_0}a_1b_2+\frac{v^*B_0^4+2v^*B_0^2+v^*}{4A_0\beta_0}a_1b_5\\&+\frac{v^*B_0^4-v^*B_0^3A_0+2v^*B_0^2+v^*A_0B_0+v^*}{8A_0\beta_0}b_2b_5+\frac{B_0^3-B_0A_0^2+B_0}{8\beta_0}b_2^2\\&-\frac{3B_0^2A_0+3A_0}{8A_0}b_4+\frac{B_0^2-2A_0B_0+1}{8}b_5.\end{aligned}$$

Define

$$\mu_2=-\frac{a(c_0)}{\alpha'(c_0)}.$$

Recall that $\alpha'(c_0)>0$ and Poincaré-Andronov-Hopf bifurcation theorem [9]. We have the following conclusion.

Theorem 2.1. Suppose that (H_0) holds, then system (2) produces a Hopf bifurcation at $c=c_0$. Furthermore,

(a) the direction of the Hopf bifurcation is subcritical and the bifurcated periodic solutions are unstable if $a(c_0)>0$;

(b) the direction of the Hopf bifurcation is supercritical and the bifurcated periodic solutions are orbitally asymptotically stable if $a(c_0)<0$.

3. Conclusion

In this paper, Hopf bifurcation of Rosenzweig-MacArthur predator-prey model with the hyperbolic tangent functional response is mainly considered. We chose capturing efficiency c of predator as the bifurcation parameter and verified the transversality condition. Moreover, the result specifically showed that $\alpha'(c_0)>0$. Then the direction and stability of Hopf bifurcation were determined by Poincaré-Andronov-Hopf bifurcation theorem. The conclusion is that the direction of the Hopf bifurcation is subcritical (resp. supercritical) and the bifurcated periodic solutions are unstable (resp. orbitally asymptotically stable) if $a(c_0)>0$ (resp. $a(c_0)<0$). It's worth noting that model (2) has richer dynamic behaviors that we'll reconsider them in the future.

Acknowledgements. We are very much thankful the editor and reviewer for their valuable comments and suggestions for the improvement of this paper. We humbly accept their suggestions and make some minor revisions.

REFERENCES

1. M.M.Haque, F.Ahmed, S.Anam and M.R.Kabir, Future population projection of Bangladesh by growth rate modeling using logistic population model, *Annals of Pure and Applied Mathematics*, 1(2) (2012) 192-202.
2. P.K.Das, Nonlinear statistical model and its applications to diffusion of mobile telephony in India, *Annals of Pure and Applied Mathematics*, 5(1) (2013) 64-70.
3. L.E.Ali, B.R.Khan and I.S.Sams, Brief study on census and predicted population of bangladesh using logistic population model, *Annals of Pure and Applied Mathematics*, 10(1) (2015) 41-47.
4. G.F.Fussmann and B.Blasius, Community response to enrichment is highly sensitive to model structure. *Biology Letters*, 1 (2005) 9-12.
5. C.S.Holling, The components of predation as revealed by a study of small-mammal predation of the European pine sawfly, *The Canadian Entomologist*, 91 (1959) 293-320.
6. V.S.Ivlev, *Experimental ecology of the feeding of fishes*, Yale University Press, New Haven, 1961.
7. A.D.Jassby and T.Platt, Mathematical formulation of the relationship between photosynthesis and light for phytoplankton, *Limnology and Oceanography*, 21 (1976) 540-547.
8. G.Seo and G.S.K.Wolkowicz, Sensitivity of the dynamics of the general Rosenzweig-MacArthur model to the mathematical form of the functional response: a bifurcation theory approach, *Journal of Mathematical Biology*, 76 (2018) 1873-1906.
9. B.D.Hassard, N.D.Kazarinoff and Y.H.Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge University Press, 1981.