Edge Regularity on m-Bipolar Fuzzy Graph

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Abstract. In this article, a new idea of m-bipolar fuzzy graph (m-BPFG) is initiated. Further, degree of an edge and total degree of an edge are defined and also determined necessary and sufficient condition under which edge regular m-BPFG and totally edge regular m-BPFG are equivalent.

Keywords: m-BPFG, Edge degree, Total edge degree.

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1. Introduction
Fuzzy sets are introduced for the parameters to solve problems related to vague and uncertain in real life situations were given by Zadeh [12] in 1965. The limitations of traditional model were overcome by the introduction of bipolar fuzzy set concept in 1994 by Zhang [13]. This was further improved by Chen et al. [4] to m-polar fuzzy set theory.

2. Preliminaries

In this section, basic terminologies of bipolar fuzzy graph (BPFG) and m-polar fuzzy graph (m-PFG) are studied.

For a given set \( V \), define an equivalence relation \( \leftrightarrow \) on \( V \times V - \{(k, k) : k \in V\} \) as follows: \((k_1, l_1) \leftrightarrow (k_2, l_2) \Leftrightarrow \) either \((k_1, l_1) = (k_2, l_2)\) or \(k_1 = l_2, l_1 = k_2\). The quotient set got in this way is denoted by \( \overline{V}^2 \).

**Definition 2.1.** A bipolar fuzzy graph of a graph \( G^b = (V, E) \) is a pair \( G = (V, S, T) \) where \( S = [\psi^p_s, \psi^o_s] \) is a bipolar fuzzy set in \( V \) and \( T = [\psi^p_t, \psi^o_t] \) is a bipolar fuzzy relation on \( \overline{V}^2 \) such that \( \psi^o_t(s, t) \leq \min \{\psi^p_s(s), \psi^o_s(t)\} \) and \( \psi^o_t(s, t) \geq \max \{\psi^o_s(s), \psi^o_s(t)\} \) for all \((s, t) \in \overline{V}^2\) and \( \psi^o_t(s, t) = 0 \) for all \((s, t) \in \overline{V}^2 - E\).

**Definition 2.2.** An m-polar fuzzy graph of a graph \( G^b = (V, E) \) is a pair \( G = (V, S, T) \) where \( S : V \rightarrow [0, 1]^m \) is an m-polar fuzzy set in \( V \) and \( T : \overline{V}^2 \rightarrow [0, 1]^m \) is an m-polar fuzzy set in \( \overline{V}^2 \) such that \( p_j \circ T(s, t) \leq \min \{p_j \circ S(s), p_j \circ S(t)\} \) for all \((s, t) \in \overline{V}^2, j = 1, 2, \cdots, m\) and \( T(s, t) = (0, 0, \cdots, 0) \) for all \((s, t) \in (\overline{V}^2 - E)\). Here, \( p_j \circ S(s) \) and \( p_j \circ T(s, t) \) represents the \( j^{th} \) component of the degree of membership value of the vertex \( 's' \) and the edge \( '(s, t)' \).

3. Regularity on m-bipolar fuzzy graphs

All the vertices and edges of an m-polar fuzzy graph have \( m \) components and these components are fixed. But these components may be bipolar. Using this idea, m-BPFG has been introduced. Before defining m-bipolar fuzzy graph, we assume the following:

**Definition 3.1.** An m-bipolar fuzzy set (m-BPFS) \( S \) on \( V \) is defined by

\[
S(s) = \left\{ \left[ p_1 \circ \psi^o_s(s), p_1 \circ \psi^p_s(s) \right], \left[ p_2 \circ \psi^o_s(s), p_2 \circ \psi^p_s(s) \right], \cdots, \left[ p_m \circ \psi^o_s(s), p_m \circ \psi^p_s(s) \right] \right\}
\]
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for all \( s \in V \) or shortly

\[
S(s) = \left\{ \left. \left[ p_j \circ \psi^p_s(s) , p_j \circ \psi^n_s(s) \right] \right| s \in V \right\}
\]

where the functions \( p_j \circ \psi^p_s : V \rightarrow [0, 1] \) and \( p_j \circ \psi^n_s : V \rightarrow [-1, 0] \) denote the positive memberships and negative memberships of the element respectively.

**Definition 3.2.** Let \( S \) be an m-BPFS on a set \( V \). An m-bipolar fuzzy relation on a set \( S \) is an m-BPFS \( T \) of \( \times V \),

\[
T(s,t) = \left\{ \left[ p_j \circ \psi^p_s(s,t) , p_j \circ \psi^p_s(s,t) \right] , \left[ p_j \circ \psi^p_s(s,t) , p_j \circ \psi^p_s(s,t) \right] , \ldots , \left[ p_j \circ \psi^p_s(s,t) , p_j \circ \psi^p_s(s,t) \right] \right\}
\]

for all \( s, t \in V \) or shortly \( T(s,t) = \left\{ \left[ p_j \circ \psi^p_s(s,t) , p_j \circ \psi^p_s(s,t) \right] \right\} \) such that

\[
p_j \circ \psi^p_s(s,t) \leq \min \{ p_j \circ \psi^p_s(s) , p_j \circ \psi^p_s(t) \}, \quad p_j \circ \psi^p_s(s,t) \geq \max \{ p_j \circ \psi^p_s(s) , p_j \circ \psi^p_s(t) \}
\]

for every \( j = 1, 2, \ldots, m \) and \( s, t \in V \).

**Definition 3.3.** An m-bipolar fuzzy graph (m-BPFG) of a graph \( G = (V, E) \) is a pair \( G = (V, S, T) \) where \( S = \left\{ p_j \circ \psi^p_s \right\} \), \( p_j \circ \psi^p_s : V \rightarrow [0, 1] \) and \( p_j \circ \psi^n_s : V \rightarrow [-1, 0] \) is an m-BPFS on \( V \); and \( T = \left\{ p_j \circ \psi^p_s \right\} \), \( p_j \circ \psi^p_s : V^2 \rightarrow [0, 1] \) and \( p_j \circ \psi^n_s : V^2 \rightarrow [-1, 0] \) is an m-BPFS in \( V^2 \) such that

\[
p_j \circ \psi^p_s(k,l) \leq \min \{ p_j \circ \psi^p_s(k) , p_j \circ \psi^p_s(l) \}, \quad p_j \circ \psi^p_s(k,l) \geq \max \{ p_j \circ \psi^p_s(k) , p_j \circ \psi^p_s(l) \}
\]

for all \( (k,l) \in V^2 \), \( j = 1, 2, \ldots, m \) and \( p_j \circ \psi^p_s(k,l) = p_j \circ \psi^p_s(k,l) = 0 \) for all \( (k,l) \in V^2 \) - \( E \).

**Example 3.1.** An example of a 3-BPFG is as shown in Figure 1.

![Figure 1: 3-Bipolar fuzzy graph G](image-url)
Definition 3.4. The degree of a vertex $r$ in an $m$-BPFG $G = (V, S, T)$ of $G^*=(V, E)$ is $d_G(r) = \left\lceil \sum_{j \in \mathbb{Z}_m} p_j \circ d_G^p(r), p_j \circ d_G^o(r) \right\rceil$ where $p_j \circ d_G^p(r) = \sum_{\{r,s\} \in E} p_j \circ \psi_T^o(r, s)$ and $p_j \circ d_G^o(r) = $.

Definition 3.5. The degree of an edge $(r, s) \in E$ in an $m$-BPFG $G = (V, S, T)$ of $G^*=(V, E)$ is $d_G(r, s) = \left\lceil \sum_{j \in \mathbb{Z}_m} p_j \circ d_G^p(r, s), p_j \circ d_G^o(r, s) \right\rceil$ where $p_j \circ d_G^p(r, s) = p_j \circ d_G^p(r) + p_j \circ d_G^p(s) - 2p_j \circ \psi_T^o(r, s),$ $p_j \circ d_G^o(r, s) = p_j \circ d_G^o(r) + p_j \circ d_G^o(s) - 2p_j \circ \psi_T^o(r, s).$

Definition 3.6. The total degree of an edge $(r, s) \in E$ in an $m$-BPFG $G = (V, S, T)$ of $G^*=(V, E)$ is $td_G(r, s) = \left\lceil \sum_{j \in \mathbb{Z}_m} p_j \circ td_G^p(r, s), p_j \circ td_G^o(r, s) \right\rceil$ where $p_j \circ td_G^p(r, s) = p_j \circ d_G^p(r) + p_j \circ d_G^p(s) - p_j \circ \psi_T^o(r, s),$ $p_j \circ td_G^o(r, s) = p_j \circ d_G^o(r) + p_j \circ d_G^o(s) - p_j \circ \psi_T^o(r, s).$

Definition 3.7. If each vertex of an $m$-BPFG $G = (V, S, T)$ of $G^*=(V, E)$ is having the same degree $\left\lceil \delta^p_j, \delta^o_j \right\rceil \in \mathbb{Z}_m$, then $G$ is called regular $m$-BPFG.

Definition 3.8. If each edge of an $m$-BPFG $G = (V, S, T)$ of $G^*=(V, E)$ is having the same degree $\left\lceil \delta^p_j, \delta^o_j \right\rceil \in \mathbb{Z}_m$, then $G$ is called an edge regular $m$-BPFG.

Definition 3.9. If each edge of an $m$-BPFG $G = (V, S, T)$ of $G^*=(V, E)$ is having the same total degree $\left\lceil \delta^p_j, \delta^o_j \right\rceil \in \mathbb{Z}_m$, then $G$ is called totally edge regular $m$-BPFG.

Theorem 3.1. Let $G = (V, S, T)$ be an $m$-BPFG on a cycle $G^*=(V, E)$. Then $\sum_{v_i \in V} d_G(v_i) = \sum_{(v_i, v_k) \in E \setminus \{e\}} d_G(v_i, v_k).$
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**Proof:** Suppose that $G = (V, S, T)$ is an m-BPFG and $G^*$ is a cycle $v_i v_2 v_3 \cdots v_i$. Then

$$\sum_{i=1} d_G(v_i, v_{i+1}) = \left( \sum_{i=1}^j p_j \circ d^e_G(v_i, v_{i+1}), \sum_{i=1}^j p_j \circ d^n_G(v_i, v_{i+1}) \right).$$

Now for $j = 1, 2, \cdots, m$

$$\sum_{i=1}^j p_j \circ d^e_G(v_i, v_{i+1}) = p_j \circ d^e_G(v_1, v_2) + p_j \circ d^e_G(v_2, v_3) + \cdots + p_j \circ d^e_G(v_i, v_1),$$

where $v_{i+1} = v_i$. Then

$$\sum_{i=1}^j p_j \circ d^e_G(v_i, v_{i+1}) - 2 \sum_{i=1}^j P_j \circ \psi^e_T(v_i, v_{i+1}) = 2 \sum_{i=1}^j p_j \circ d^e_G(v_i, v_{i+1}).$$

Similarly,

$$\sum_{i=1}^j P_j \circ d^e_G(v_i, v_{i+1}) - 2 \sum_{i=1}^j P_j \circ \psi^e_T(v_i, v_{i+1}) = \sum_{i=1}^j P_j \circ d^e_G(v_i).$$

**Remark 3.1.** Let $G = (V, S, T)$ be an m-BPFG on $G^* = (V, E)$. Then

$$\sum_{(v_i, v_k) \in E} d_G(v_i, v_k) = \left( \sum_{(v_i, v_k) \in E} d_G(v_i, v_k), \sum_{(v_i, v_k) \in E} d_G(v_i, v_k) \right).$$

where $d_G(v_i, v_k) = d_G(v_i) + d_G(v_k) - 2 \text{ for all } (v_i, v_k) \in E$.

**Theorem 3.2.** Let $G = (V, S, T)$ be an m-BPFG on a $c$-regular graph $G^* = (V, E)$. Then

$$\sum_{(v_i, v_k) \in E} d_G(v_i, v_k) = (c-1) \sum_{v_i \in V} d_G(v_i).$$

**Proof:** From Remark 3.1., we have

$$\sum_{(v_i, v_k) \in E} d_G(v_i, v_k) = \left( \sum_{(v_i, v_k) \in E} d_G(v_i, v_k) p_j \circ \psi^e_T(v_i, v_k), \sum_{(v_i, v_k) \in E} d_G(v_i, v_k) p_j \circ \psi^e_T(v_i, v_k) \right)$$

$$= \left( \sum_{(v_i, v_k) \in E} (d_G(v_i) + d_G(v_k) - 2p_j \circ \psi^e_T(v_i, v_k)), \sum_{(v_i, v_k) \in E} (d_G(v_i) + d_G(v_k) - 2p_j \circ \psi^e_T(v_i, v_k)) \right).$$
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Since $G^*$ is a regular graph, we have the degree of every vertex in $G^*$ is $c$. So

$$
\sum_{(v_i, v_j) \in E} d_G(v_i, v_j) = (c + c - 2) \left( \sum_{(v_i, v_j) \in E} p_j \circ \psi_T^m(v_i, v_j) \right) + \sum_{(v_i, v_j) \in E} p_j \circ \psi_T^m(v_i, v_j)
$$

$$
\sum_{(v_i, v_j) \in E} d_G(v_i, v_j) = 2(c - 1) \left( \sum_{(v_i, v_j) \in E} p_j \circ \psi_T^m(v_i, v_j) \right) + \sum_{(v_i, v_j) \in E} p_j \circ \psi_T^m(v_i, v_j)
$$

$$
\sum_{(v_i, v_j) \in E} d_G(v_i, v_j) = (c - 1) \sum_{v_j \in V} d_G(v_j).
$$

**Theorem 3.3.** Let $G = (V, S, T)$ be an m-BPFG on a crisp graph $G^* = (V, E)$. Then,

$$
\sum_{(v_i, v_j) \in E} td_G(v_i, v_j) = \left( \sum_{(v_i, v_j) \in E} d_G(v_i, v_j) \right) + \sum_{(v_i, v_j) \in E} \left( p_j \circ \psi_T^m(v_i, v_j) \right)^m
$$

**Proof:** From the definition of total edge degree, we have

$$
\sum_{(v_i, v_j) \in E} td_G(v_i, v_j) = \sum_{(v_i, v_j) \in E} \left( d_G(v_i, v_j) + \left( p_j \circ \psi_T^m(v_i, v_j) \right)^m \right)
$$

$$
= \sum_{(v_i, v_j) \in E} d_G(v_i, v_j) + \sum_{(v_i, v_j) \in E} \left( p_j \circ \psi_T^m(v_i, v_j) \right)^m
$$

From Remark 3.1., we have

$$
\sum_{(v_i, v_j) \in E} td_G(v_i, v_j) = \left( \sum_{(v_i, v_j) \in E} d_G(v_i, v_j) \right) + \sum_{(v_i, v_j) \in E} \left( p_j \circ \psi_T^m(v_i, v_j) \right)^m
$$

**Theorem 3.4.** Let $G = (V, S, T)$ be an m-BPFG on a crisp graph $G^* = (V, E)$. Then

$$
T = \left[ p_j \circ \psi_T^m, p_j \circ \psi_T^m \right]_{j=1}^m
$$

is a constant function if and only if the subsequent conditions are equivalent:

(i) $G$ is an edge regular m-BPFG.

(ii) $G$ is a totally edge regular m-BPFG.

**Proof:** Let us suppose that $T$ be a constant function.

Then $\left[ p_j \circ \psi_T^m(\alpha, \beta), p_j \circ \psi_T^m(\alpha, \beta) \right]_{j=1}^m = \left[ p_j^m, p_j^m \right]_{j=1}^m$ for all $(\alpha, \beta) \in E$, where
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\( \gamma_0 \in [0, 1], \gamma_1 \in [-1, 0] \). Let \( G \) be an edge regular m-BPFG. Then for all \( (v_i, v_j) \in E \),
\[
d_G(v_i, v_j) = \left[ \delta_i^n, \delta_j^n \right]_{j=1}^m.
\] Now we prove that \( G \) is a totally edge regular m-BPFG.

Now we prove that \( G \) is a totally edge regular m-BPFG.

\[
td_G(v_i, v_j) = d_G(v_i, v_j) + \left[ p_j \circ \psi_{T}^p (v_i, v_j), p_j \circ \psi_{T}^n (v_i, v_j) \right]_{j=1}^m = \left[ h_i^n, h_j^n \right]_{j=1}^m.
\]

Now we prove that \( G \) is a totally edge regular m-BPFG.

Hence,
\[
d_G(v_i, v_j) = \left[ h_i^n, h_j^n \right]_{j=1}^m - \left[ h_i^n - \gamma_i^n, h_j^n - \gamma_j^n \right]_{j=1}^m
\]

for all \( (v_i, v_j) \in E \). Then \( G \) is an \( \left[ h_i^n - \gamma_i^n, h_j^n - \gamma_j^n \right]_{j=1}^m \)-edge regular m-BPFG.

Conversely, we assume that conditions (i) and (ii) are equivalent. Now we have to show that the function \( T \) is constant. In a contrary way suppose that, the function \( T \) is not constant. Then \( \left[ p_j \circ \psi_{T}^p (v_i, v_j), p_j \circ \psi_{T}^n (v_i, v_j) \right]_{j=1}^m \neq \left[ p_j \circ \psi_{T}^p (v_i, v_j), p_j \circ \psi_{T}^n (v_i, v_j) \right]_{j=1}^m \) for at least one pair of edges \( (v_i, v_j), (v_i, v_s) \in E \). Let \( G \) be a \( \left[ \delta_i^n, \delta_j^n \right]_{j=1}^m \)-edge regular m-BPFG. Then \( d_G(v_i, v_j) = d_G(v_i, v_s) = \left[ \delta_i^n, \delta_j^n \right]_{j=1}^m \). Then for \( (v_i, v_j), (v_i, v_s) \in E \), we have
\[
td_G(v_i, v_j) = d_G(v_i, v_j) + \left[ p_j \circ \psi_{T}^p (v_i, v_j), p_j \circ \psi_{T}^n (v_i, v_j) \right]_{j=1}^m
\]
and
\[
td_G(v_i, v_s) = d_G(v_i, v_s) + \left[ p_j \circ \psi_{T}^p (v_i, v_s), p_j \circ \psi_{T}^n (v_i, v_s) \right]_{j=1}^m
\]

for all \( (v_i, v_j) \in E \). Then \( G \) is an \( \left[ \delta_i^n, \delta_j^n \right]_{j=1}^m \)-edge regular m-BPFG.
we have \( td_G(v_i, v_j) \neq td_G(v_s, v_t) \). Hence \( G \) is not a totally edge regular m-BPFG.

This is a contradiction to our assumption and so that the function \( T \) is constant.

Similarly, we can show that the function \( T \) is constant, when \( G \) is a totally edge regular m-BPFG.

**Theorem 3.5.** Let \( G^* = (V, E) \) be a \( h \)-regular crisp graph and \( G = (V, S, T) \) be an m-BPFG on \( G^* \). Then, the function \( T = \left( \left\{ p_j \circ \psi^h_{\alpha} \right\}_{j=1}^{m}, \left\{ p_j \circ \psi^h_{\beta} \right\}_{j=1}^{m} \right) \) is constant if and only if \( G \) is both regular m-BPFG and totally edge regular m-BPFG.

**Proof:** Let \( T \) be a constant function. Then \( \left( \left\{ p_j \circ \psi^h_{\alpha} \right\}_{j=1}^{m}, \left\{ p_j \circ \psi^h_{\beta} \right\}_{j=1}^{m} \right) \) for all \( \left( \alpha, \beta \right) \in E \) where \( \gamma^o_j \) and \( \gamma^o_j \) are constants. From the definition of degree of a vertex, we get

\[
d_G(v_i) = \left\{ \sum_{(v_i, v_j) \in E} p_j \circ \psi^h_{\alpha} (v_i, v_j), \sum_{(v_i, v_j) \in E} p_j \circ \psi^h_{\beta} (v_i, v_j) \right\} = \left\{ \sum_{j=1}^{m} \gamma^o_j, \sum_{j=1}^{m} \gamma^o_j \right\} = \left\{ h \gamma^o_j, h \gamma^o_j \right\}_{j=1}^{m} \ 	ext{for all } v_i \in V.
\]

Therefore, \( G \) is a regular m-BPFG.

Again,

\[
td_G(v_i, v_j) = \left\{ \sum_{(v_i, v_j) \in E} p_j \circ \psi^h_{\alpha} (v_i, v_j), \sum_{(v_i, v_j) \in E} p_j \circ \psi^h_{\beta} (v_i, v_j) \right\} = \left\{ \sum_{j=1}^{m} \gamma^o_j, \sum_{j=1}^{m} \gamma^o_j \right\} = \left\{ h \gamma^o_j, h \gamma^o_j \right\}_{j=1}^{m} \ 	ext{for all } (v_i, v_j) \in E.
\]
Conversely, assume that \( G \) is both regular and totally edge regular m-BPFG. Now we have to prove that \( T \) is a constant function. Since \( G \) is regular, 
\[
d_G(v_l) = \left[ z_j^p, z_j^n \right]_{j=1}^m
\]
for all \( v_l \in V \). Also \( G \) is totally edge regular. Hence, 
\[
td_G(v_l, v_r) = \left[ h_j^p, h_j^n \right]_{j=1}^m
\]
for all \((v_l, v_r) \in E\). From the definition of totally edge degree, we get 
\[
td_G(v_l, v_r) = \left[ p_j \circ d_G^p(v_l), p_j \circ d_G^p(v_r) \right]_{j=1}^m + \left[ p_j \circ d_G^n(v_l), p_j \circ d_G^n(v_r) \right]_{j=1}^m
\]
- \left[ p_j \circ \psi_T^p(v_l, v_r), p_j \circ \psi_T^p(v_l, v_r) \right]_{j=1}^m
\]
for all \((v_l, v_r) \in E\). Hence \( T \) is a constant function.

4. Conclusions
In this article, edge degree and total edge degree of an m-BPFG are defined. Further, an equivalence condition for edge regular m-BPFG and totally edge regular m-BPFG is given. In future we intend to extend our work to density of m-BPFG and morphism between two m-BPFGs and study some of its properties.

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REFERENCES