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# The Existence of Positive Solutions for Third-Order Three-Point Nonhomogeneous Boundary Value Problems

***Jiali Da***

Department of Mathematics
Zhixing College of Northwest Normal University
GanSu Lanzhou, 730070, China.
Email：335855313@qq.com.

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***Abstract.*** In this paper, we investigate the existence of positive solutions for third-order three-point nonhomogeneous boundary value problems. By using Leray-Schauder fix point theorem, some sufficient conditions for the existence of positive solutions are obtained, which improve the previous results.

***Keywords:*** Nonhomogeneous boundary value problems, Positive solution, Leray-Schauder equation

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## 1. Introduction

In latest years, third-order differential equations appear in various fields of applied mathematics and physics, and third-order three-point boundary value problems have always been the focus of attention. One may see Anderson [1,2], Anderson and Davis [3], Bai [4], Boucherif and Al-Malki [5]. Since then, Tariboon and Sitthiwirattham [6] studied the three-point integral boundary value problem. Recently, all sorts of three-point boundary value problems for nonlinear differential equations have been studied by many authors. We refer the readers to [7,8,9,10,11].

In 2009, Sun [7] studied the existence of positive solutions of the following third-order three-point inhomogeneous boundary value problem

$$\begin{cases} u'''(t)+a(t)f(u(t))=0, \quad t\in(0,1) & (1.1)\\ u(0)=u'(0)=0, \quad u'(1)-\alpha u'(\eta)=\lambda & (1.2)\end{cases}$$

where $\eta\in(0,1)$, $\alpha\in[0,\frac{1}{\eta})$ are constants and $\lambda\in(0,\infty)$ is a parameter.

Assume that:

$(A_1)$ $f \in C([0,\infty),[0,\infty))$;

$(A_2)$ $a \in C([0,1],[0,\infty))$, $and$ $0 < \int_0^1 (1-s)sa(s)ds < \infty$.

Let

$$f_0 = \lim_{u^+ \to \infty} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}$$

In the article [12] studied the existence of positive solutions of boundary value problems (1.1)-(1.2), by using Krasnoselskii fixed point theorem, the obtained the following results:

**Theorem 1.1.** If *f* satisfies the superlinear condition: then the boundary value problem (1.1)-(1.2) has at least one positive solution when it is sufficiently small and there is no positive solution when it is sufficiently large.

**Theorem 1.2.** If f satisfies the sublinear condition: then the boundary value problem (1.1)-(1.2) must have at least one positive solution.

In this article, we discuss the existence of positive solutions for boundary value problems (1.1)-(1.2). Using the Leary-Schauder fixed point theorem, our results are better than the conditions of Theorem 1.1 and Theorem 1.2.

The rest of this paper is organized as follows. In section 2, we present some preliminaries that will be used in Section 3.The main results and proofs will be given in Section 3.Finally,the conclusion and future work are given in Section 4.

## 2. Preliminaries

Consider the boundary value problem

$$\begin{cases} u'''(t) + p(t) = 0, & t \in (0,1) \quad (2.1) \\ u(0) = u'(0) = 0, & u'(1) - \alpha u'(\eta) = \lambda \quad (2.2) \end{cases}$$

**Lemma 2.1.** [12] Let $x \in C^+[0,1] := \{x \in C[0,1], x(t) \geq 0, t \in [0,1]\}$. Then the problem (2.1)-(2.2) has a unique solution

$$u(t) = \int_0^1 G(t,s)p(s)ds + \frac{\alpha t^2}{2(1-\alpha\eta)} \int_0^1 G_1(\eta,s)p(s)ds + \frac{\lambda t^2}{2(1-\alpha\eta)}$$

where $$G(t,s)=\frac{1}{2}\begin{cases}(2t-t^2-s)s, & s\le t\\ (1-s)t^2, & t\le s\end{cases}$$

and $$G_1(t,s)=\frac{\partial G(t,s)}{\partial t}=\begin{cases}(1-t)s, & s\le t\\ (1-s)t, & t\le s\end{cases}$$

**Lemma 2.2.** [12] Let $(t,s)\in[0,1]\times[0,1]$, then $0\le G_1(t,s)\le(1-s)s$.

**Lemma 2.3.** [12] Let $(t,s)\in[0,1]\times[0,1]$,

then $$\gamma G(1,s)\le G(t,s)\le G(1,s)=\frac{1}{2}(1-s)s,$$

where $\gamma=\tau^2$, and satisfies $\int_\tau^1(1-s)sa(s)ds>0$.

**Lemma 2.4.** [12] If $x\in C^+[0,1]$ then the only solution of problems (2.1)-(2.2) is non-negative and satisfies

$$\min_{t\in[\tau,1]}u(t)\ge\gamma\|u\|$$

For any $y(t)\in C[0,1]$, Consider the problem

$$\begin{cases}u'''(t)+a(t)f(u(t))=0, & t\in(0,1)\\ u(0)=u'(0)=0, & \mathrm{u}'(1)-\alpha u'(\eta)=\lambda\end{cases}$$

From Lemma 2.1, (1.1)-(1.2) has a unique solution

$$u(t)=\int_0^1 G(t,s)a(s)f(y(s))ds+\frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 G_1(\eta,s)a(s)f(y(s))ds+\frac{\lambda t^2}{2(1-\alpha\eta)}.$$

Definition operator

$$Ty(t)=\int_0^1 G(t,s)a(s)f(y(s))ds+\frac{\alpha t^2}{2(1-\alpha\eta)}\int_0^1 G_1(\eta,s)a(s)f(y(s))ds+\frac{\lambda t^2}{2(1-\alpha\eta)}.$$

Obviously, y(t) is the solution of the boundary value problem (1.1)-(1.2) if and only if y(t) is the fixed point of the operator T.

**Lemma 2.5.** [12] **(**Leray-Schauder) Let $\Omega$be the convex subset of Banah space X, $00\in\Omega,\phi:\Omega\to\Omega$ be complete continuous operator, then

(1) $\phi$ has at least one fixed point in $\Omega$;

(2) $\{x \in C | x = \lambda\Phi x, 0 < \lambda < 1\}$ is unbounded.

### 3. Main results

Let
$$X = C^{+}[0,1], \qquad \beta = \int_0^1 (1-s)sa(s)ds.$$

**Theorem 3.1.** Assume (H) hold, if $f_0 = 0$, when $\lambda \in (0, B(1-\alpha\eta))$, then the boundary value problem (1.1)-(1.2) has at least one positive solution.

**Proof**: Choose $\varepsilon > 0$ *and* $\varepsilon \le \dfrac{1-\alpha\eta}{(1-\alpha\eta+\alpha)\beta}$, By $f_0 = 0$, we know there exists constant $B > 0$, such that, $f(y) \le \varepsilon y$, for $0 < y \le B$

Let
$$\Omega = \{y | y \in C^{+}[0,1], y \ge 0, ||y|| \le B, \min_{t\in[r,1]} y(t) \ge \gamma\, ||y||\},$$

Then$\Omega$is the convex subset of X. For $y \in \Omega$, by Lemmas 2.1 and 2.4, we know
$$Ty(t) \ge 0 \text{ and } \min_{t\in[r,1]} y(t) \ge \gamma\, \big||Ty|\big|,$$

On the other hand,
$$\begin{aligned}
Ty(t) &\le \int_0^1 \frac{1}{2}(1-s)a(s)f(y(s))ds + \frac{\alpha}{2(1-\alpha\eta)}\int_0^1 (1-s)sa(s)f(y(s))ds + \frac{\lambda}{2(1-\alpha\eta)} \\
&\le \frac{1-\alpha\eta+\alpha}{2(1-\alpha\eta)}\int_0^1 (1-s)sa(s)f(y(s))ds + \frac{\lambda}{2(1-\alpha\eta)} \\
&\le \frac{\|y\|}{2} + \frac{\lambda}{2(1-\alpha\eta)} \le \frac{B}{2} + \frac{B}{2} = B
\end{aligned}$$

Thus, $\|Ty\| \le B$. Hence, $T\Omega \subset \Omega$, $T : \Omega \to \Omega$ is completely continuous.

For $y \in \Omega$ *and* $y = \lambda Ty$, $0 < \lambda < 1$, we have $y(t) = \lambda Ty(t) < Ty(t) \le B$, which implies $||y|| \le$ B.

$\{y \in \Omega \,|\, y = \lambda Ty,\ 0 < \lambda < 1\}$ is bounded. By Lemma 2.5, we know the operator T has at least one fixed point in $\Omega$ .Thus the boundary value problem (1.1)-(1.2) has at least one positive solution. The proof is complete.

**Theorem 3.2.** If $f_\infty = 0$, when $\lambda \in (0, \frac{2B(1-\alpha\eta)}{3})$, then the boundary value problem (1.1)-(1.2) has at least one positive solution。

**Proof:** $\varepsilon > 0$ *and* $\varepsilon \le \frac{2(1-\alpha\eta)}{3\beta(1-\alpha\eta+\alpha)}$. By $f_\infty = 0$, we know there exists constant $N > 0$, such that, $f(y) \le \varepsilon y$, for $y > N$

Choose

$$B \ge N + 1 + \frac{\beta(1-\alpha\eta+\alpha)}{6(1-\alpha\eta)} \max_{0\le y\le N} f(y)$$

Let

$$\Omega = \{y | y \in C^+[0,1], y \ge 0, ||y|| \le B, \min_{t\in[r,1]} y(t) \ge \gamma\, ||y||\},$$

then $\Omega$ is the convex subset of X. For $y \in \Omega$, by Lemmas 2.1 and 24, we know

$$Ty(t) \ge 0 \text{ and } \min_{t\in[r,1]} y(t) \ge \gamma\, \big||Ty|\big|.$$

On the other hand,

$$\begin{aligned}
Ty(t) &\le \frac{1-\alpha\eta+\alpha}{2(1-\alpha\eta)} \int_0^1 (1-s)sa(s)f(y(s))ds + \frac{\lambda}{2(1-\alpha\eta)} \\
&= \frac{1-\alpha\eta+\alpha}{2(1-\alpha\eta)} \Big(\int_{J_1=\{s\in[0,1], y(s)>N\}} (1-s)sa(s)f(y(s))ds \\
&\quad + \int_{J_2=\{s\in[0,1], y(s)\le N\}} (1-s)sa(s)f(y(s))ds\Big) + \frac{\lambda}{2(1-\alpha\eta)} \\
&\le \frac{1-\alpha\eta+\alpha}{2(1-\alpha\eta)} \int_0^1 (1-s)sa(s)\varepsilon y(s)ds + \frac{1-\alpha\eta+\alpha}{2(1-\alpha\eta)} \int_0^1 (1-s)sa(s)ds \max\nolimits_{0\le y\le N} f(y) + \frac{\lambda}{2(1-\alpha\eta)} \\
&\le \frac{B}{3} + \frac{B}{3} + \frac{B}{3} = B.
\end{aligned}$$

Thus, $\| Ty \| \le B$. Hence, $T\Omega \subset \Omega$, $T: \Omega \to \Omega$ is completely continuous.
For $y \in \Omega$ *and* $y = \lambda Ty$, $0 < \lambda < 1$, we have $y(t) = \lambda Ty(t) < Ty(t) \le B$, which implies $\| y \| \le B$.

$\{y \in \Omega \,|\, y = \lambda Ty,\ 0 < \lambda < 1\}$ is bounded. By Lemma 2.5, we know the operator T has at least one fixed point in $\Omega$, Then the boundary value problem (1.1)-(1.2) has at least one positive solution.

**Theorem 3.3.** Assume $(H)$ hold, if there exists constant $\rho_1 > 0$, when $\lambda \in (0, \rho_1(1-\alpha\eta))$, such that, $f(y) \le \frac{(1-\alpha\eta)\rho_1}{(1-\alpha\eta+\alpha)\beta}$, $for\ 0 < y \le \rho_1$, then the boundary value problem (1.1)-(1.2)has at least one positive solution.

**Proof:** Let

$$\Omega = \{y | y \in C^+[0,1], y \ge 0, ||y|| \le \rho_1, \min_{t\in[r,1]} y(t) \ge \gamma\ ||y||\},$$

then $\Omega$ is the convex subset of X， for $y \in \Omega$, by Lemmas 2.1 and 2.4, we know

$$Ty(t) \ge 0 \text{ and } \min_{t\in[r,1]} Ty(t) \ge \gamma\ \ ||Ty||.$$

On the other hand,

$$Ty(t) \le \frac{1-\alpha\eta+\alpha}{2(1-\alpha\eta)} \int_0^1 (1-s)sa(s)f(y(s))ds + \frac{\lambda}{2(1-\alpha\eta)} \le \frac{\rho_1}{2} + \frac{\rho_1}{2} = \rho_1.$$

Thus, $\| Ty \| \le \rho_1$. *Hence,* $T\Omega \subset \Omega$, $T : \Omega \to \Omega$ is completely continuous.

For $y \in \Omega$ *and* $y = \lambda Ty$, $0 < \lambda < 1$, we have $y(t) = \lambda Ty(t) < Ty(t) \le \rho_1$, $\| y \| \le \rho_1$.

So, $\{y \in \Omega \mid y = \lambda Ty,\ 0 < \lambda < 1\}$ is bounded. By Lemma 2.5，we know the operator T has at least one fixed point in $\Omega$. Thus the boundary value problem (1.1)-(1.2) has at least one positive solution.

**Theorem 3.4.** Assume $(H)$ hold. If there exists constant $\rho_2 > 0$, such that

$$f(y) \le \frac{(2-\alpha\eta^2)\rho_2}{2\beta}, \quad for\ y \ge \rho_2,$$

then the boundary value problem (1.1)-(1.2) has at least one positive solution.

**Proof:** Choose

$$d \ge 1 + \rho_2 + \frac{\beta(1-\alpha\eta+\alpha)}{2(1-\alpha\eta)} \max_{0\le y\le\rho_2} f(y) + \frac{\lambda}{2(1-\alpha\eta)}$$

Let

$$\Omega = \{y | y \in C^{+}[0,1], y \geq 0, ||y|| \leq d, \min_{t\in[r,1]} y(t) \geq \gamma\, ||y||\},$$

then $\Omega$ is the convex subset of X. For $y \in \Omega$, by Lemmas 2.1 and 2.4， we know

$$Ty(t) \geq 0 \text{ and } \min_{t\in[r,1]} Ty(t) \geq \gamma\ ||Ty||.$$

On the other hand,

$$\begin{aligned} Ty(t) &\leq \frac{1-\alpha\eta+\alpha}{2(1-\alpha\eta)} \int_0^1 (1-s)sa(s) f(y(s))ds + \frac{\lambda}{2(1-\alpha\eta)} \\ &= \frac{1-\alpha\eta+\alpha}{2(1-\alpha\eta)} \Big( \int_{J_1=\{s\in[0,1], y(s)>\rho_2\}} (1-s)sa(s) f(y(s))ds \\ &\quad + \int_{J_2=\{s\in[0,1], y(s)\leq\rho_2\}} (1-s)sa(s) f(y(s))ds \Big) + \frac{\lambda}{2(1-\alpha\eta)} \end{aligned}$$

$$\begin{aligned} &\leq \frac{1-\alpha\eta+\alpha}{2(1-\alpha\eta)} \int_0^1 (1-s)sa(s) \frac{2(1-\alpha\eta)\rho_2}{\beta(1-\alpha\eta+\alpha)} ds \\ &\quad + \frac{1-\alpha\eta+\alpha}{2(1-\alpha\eta)} \int_0^1 (1-s)sa(s)ds \max_{0\leq y\leq\rho_2} f(y) + \frac{\lambda}{2(1-\alpha\eta)} \\ &= \rho_2 + \frac{1-\alpha\eta+\alpha}{2(1-\alpha\eta)} \beta \max_{0\leq y\leq \rho 2} f(y) + \frac{\lambda}{2(1-\alpha\eta)} < d. \end{aligned}$$

Thus, $||Ty|| \leq d$. Hence, $T\Omega \subset \Omega$, $T:\Omega \to \Omega$ is completely continuous.

For $y \in \Omega$ *and* $y = \lambda Ty$, $0 < \lambda < 1$, we have $y(t) = \lambda Ty(t) < Ty(t) \leq d$, $\| y \| \leq d$. So, $\{y \in \Omega \mid y = \lambda Ty,\ 0 < \lambda < 1\}$ is bounded. By Lemma 2.5，we know the operator T has at least one fixed point in $\Omega$，Thus the boundary value problem (1.1)-(1.2) has at least one positive solution.

**5. Conclusion and future work**

The main purpose of this paper was to present new result of positive solutions for third-order three-point nonhomogeneous boundary value problems with Leray-Schauder fixed point theorem, highlighting the method which has been used by Sun [12] to some results.

In the future research, the third-order boundary value problem of the scientific research award was raised to the n-order boundary value problem, and its different boundary conditions were changed to obtain the existence of its positive solution, and the existence of multiple solutions can even be considered. These suggestions will be treated in the future.

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