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Numerical Solution for Real Double Definite Integrals

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Abstract. The double integral is numerically evaluated in this paper using higher precision quadrature rules. With the combination of Newtonian and Gaussian rules of precision three each, a mixed quadrature rule of precision five is obtained. Three test problems are used to numerically validate the rule. The approximations are compared to analytical solutions, and error bounds are calculated.

Keywords: Degree of precision, Error bound, Maclaurin's theorem, Mixed quadrature rule.

AMS Mathematics Subject Classification (2010): 65D30, 65D32

1. Introduction

A mixed quadrature rules of higher degree of precision for real and complex analytic functions for single integrals are constructed by [1-4, 6-7, 11, 13, 14, 16-24, 28-30, 33-37]. A mixed quadrature rule of degree of precision-5 for double integrals is incorporated taking the convex combination of Simpson's $\frac{3}{8}th$ th and Gauss-Legendre-2 point rule each ofdegree of precision 3.The other techniques [5, 8-10, 12, 15, 25, 26, 31, 32,38-42] are the back bone to the present method.The main aim of this paper is the const ruction of mixed quadrature rule of higher degree precision of double integrals for two va riables. The aim of this work is how to implement mixed quadrature in line integral, surfa ce integral and also in volume integral in mathematical physics and electromagnetic field theory.

This paper is designed as follows. Section 1 is an introduction part. Section 2 contains construction of quadrature of constituent rules and the corresponding errors in-2 variables are obtained in Section 3. Section 4 is devoted to construction of mixed quadrature rule. The error analysis is done in section 5. In section 6 the rule is numerically verified by taking three examples. The conclusions are drawn in section 7.

2. Construction of quadrature rules in two variables

Newtonian and Gaussian quadrature are:

The Simpson's $\frac{3}{8}th$ rule is

$$I_{S_{\frac{3}{8}}}(f)=\frac{1}{16}\left[\begin{array}{l}\{f(-1,-1)+f(-1,1)+f(1,-1)+f(1,1)\}\\+3\left\{\begin{array}{l}f\left(-1,-\frac{1}{3}\right)+f\left(-1,\frac{1}{3}\right)+f\left(1,-\frac{1}{3}\right)+f\left(1,\frac{1}{3}\right)\\+f\left(-\frac{1}{3},-1\right)+f\left(-\frac{1}{3},1\right)+f\left(\frac{1}{3},-1\right)+f\left(\frac{1}{3},1\right)\end{array}\right\}\\+9\left\{f\left(-\frac{1}{3},-\frac{1}{3}\right)+f\left(-\frac{1}{3},\frac{1}{3}\right)+f\left(\frac{1}{3},-\frac{1}{3}\right)+f\left(\frac{1}{3},\frac{1}{3}\right)\right\}\end{array}\right] \tag{2.1}$$

Gauss-Legendre – 2-point rule is

$$I_{GL2}(f)=f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)+f\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) \tag{2.2}$$

$$I(f)=I_{S_{\frac{3}{8}}}(f)+E_{S_{\frac{3}{8}}}(f) \tag{2.3}$$

$$I(f)=I_{GL2}(f)+E_{GL2}(f) \tag{2.4}$$

where $E_{S_{\frac{3}{8}}}(f)$ and $E_{GL2}(f)$ are error in approximating the integrals $I(f)$ by equation (2.3) and equation (2.4) respectively.

Now assuming $f(x,y)$ to be sufficiently differentiable in $-1\le x,y\le 1$.

For approximate evaluation of real definite integral and applying Maclaurin series

$$I(f)=\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dxdy =$$

$$4f_{0,0}(0,0)+\frac{2}{3}\left[f_{2,0}(0,0)+f_{0,2}(0,0)\right]+\frac{1}{30}\left[f_{4,0}(0,0)+f_{0,4}(0,0)\right]+\frac{1}{9}f_{2,2}(0,0)$$
$$+\frac{1}{180}\left[f_{4,2}(0,0)+f_{2,4}(0,0)\right]+\frac{4}{7!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+\ldots.. \tag{2.5}$$

Applying, Maclaurin's expansion in equation (2.1)

$$I_{s_{\frac{3}{8}}}(f)=4f_{0,0}(0,0)+\frac{2}{3}\left\{f_{2,0}(0,0)+f_{0,2}(0,0)\right\}+\frac{1}{18}\left\{f_{4,0}(0,0)+f_{0,4}(0,0)\right\}$$
$$+\frac{1}{9}f_{2,2}(0,0)+\frac{3904}{243\times 6!}\left\{f_{6,0}(0,0)+f_{0,6}(0,0)\right\} \tag{2.6}$$
$$+\frac{7}{27\times 36}\left\{f_{4,2}(0,0)+f_{2,4}(0,0)\right\}\ldots.$$

Also using Maclaurin's expansion in equation (2.2)

$$I_{GL2}(f)=4f(0,0)+\frac{2}{3}\left[f_{2,0}(0,0)+f_{0,2}(0,0)\right]+\frac{1}{9\times 3!}\left[f_{4,0}(0,0)+f_{0,4}(0,0)\right]$$
$$+\frac{20}{9\times 6!}\left[f_{4,2}(0,0)+f_{2,4}(0,0)\right]+\frac{1}{9}f_{2,2}(0,0)+\frac{4}{27\times 6!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+\ldots. \tag{2.7}$$

3. Errors

In Simpson's $\frac{3}{8}th$ rule:

Using equation (2.5) and equation (2.6) in equation (2.3) error associated with Simpson's $\frac{3}{8}$th rule is

$$E_{S\frac{3}{8}}(f) = -\frac{4}{405}\{f_{4,0}(0,0) + f_{0,4}(0,0)\} - \frac{2}{1215}\{f_{4,2}(0,0) + f_{2,4}(0,0)\} - \frac{26356}{243\times 7!}\{f_{6,0}(0,0) + f_{0,6}(0,0)\} \tag{2.8}$$

In Gauss-Legendre-2 point rule:

Now the error associated with the Gauss- Legendre-2 point rule is obtained substituting equation (2.5) and (2.7) in equation (2.4)

$$E_{GL2}(f) = I(f) - R_{GL2}(f) =$$

$$\frac{2}{135}\left[f_{4,0}(0,0) + f_{0,4}(0,0)\right] + \frac{1}{405}\left[f_{4,2}(0,0) + f_{2,4}(0,0)\right] + \frac{1}{27\times 63}\left[f_{6,0}(0,0) + f_{0,6}(0,0)\right] + ... \tag{2.9}$$

Thus, the degree of the precision for each rule is 3.

4. Mixed quadrature rule

Now multiplying (2) by equation (2.8) and (3) by (2.9) and then adding them, we get

$$I(f) = \frac{1}{5}\left[2R_{S\frac{3}{8}}(f) + 3I_{GL2}(f)\right] + \frac{1}{5}\left[2E_S(f) + 3E_{GL2}(f)\right] \tag{3.1}$$

$$I_{S\frac{3}{8}GL2}(f) = \frac{1}{5}\left[2R_{S\frac{3}{8}}(f) + 3I_{GL2}(F)\right] \tag{3.2}$$

where $I_{S\frac{3}{8}GL2}(f)$ and $E_{S\frac{3}{8}GL2}(f)$ are mixed quadrature rule and its error obtained by Simpson's $\frac{3}{8}$ th and Gauss-Legendre-2 point rule respectively.

5. Error analysis

Theorem 1. Let $f(x, y)$ be sufficiently differentiable function in the closed interval $[-1,1]$. The bounds of truncation error $E_{S\frac{3}{8}GL2}(f)$ associated with the rule $I_{S\frac{3}{8}GL2}(f)$ is

$$\left|E_{S\frac{3}{8}GL2}(f)\right| \cong \frac{7}{3^5\times 400}\left|f_{6,0}(0,0) + f_{0,6}(0,0)\right|$$

Proof: The proof obviously follows from the equation (2.8).

Theorem 2. The bounds for the truncation error $\left|E_{S\frac{3}{8}GL2}(f)\right| \le \frac{4R}{675}\left|\beta_2 - \beta_1\right|$

where $\beta_1, \beta_2 \in [-1,1]$ and

$$R = \underset{\substack{-1\le x\le 1\\ -1\le y\le 1}}{Max}\left| f_{4,0}(0,0) + f_{0,4}(0,0)\right|$$

Proof: We have

$$E_{S\frac{3}{8}}(f) = -\frac{4}{405}\left| f_{4,0}(\beta_1) + f_{0,4}(\beta_1)\right|$$

$$E_{GL2}(f) = \frac{2}{135}\left| f_{4,0}(\beta_2) + f_{0,4}(\beta_2)\right|$$

Hence

$$E_{S\frac{3}{8}GL2}(f) = \frac{1}{5}\left|3E_{s\frac{3}{8}}(f) + 2E_{GL2}(f)\right|$$

$$= \frac{4}{675}\left[f_{4,0}(\beta_2,0) + f_{0,4}(0,\beta_2) - f_{4,0}(\beta_1,0) - f_{0,4}(0,\beta_1)\right]$$

$$= \frac{4}{675}\left[\int_{\beta_1}^{\beta_2} f_{5,0}(x,0)\,dx + \int_{\beta_1}^{\beta_2} f_{0,5}(0,y)\,dy \right]$$

$$= \frac{4}{675(\beta_2 - \beta_1)}\int_{\beta_1}^{\beta_2}\int_{\beta_1}^{\beta_2}\left[f_{5,0}(x,*) + f_{0,5}(*,y)\right] dxdy$$

Hence

$$\left|E_{S\frac{3}{8}GL2}(f)\right| \le \frac{4R\left|\beta_1 - \beta_2\right|}{675}$$

which, gives only the truncation error bound on β_1, β_2 are known points in [-1, 1]

Corollary 1. The error bound for the truncation error

$$\left|E_{S\frac{3}{8}GL2}(f)\right| \le \frac{8R}{675}$$

when $\left|\beta_1 - \beta_2\right| \le 2$ [27].

6. Numerical verification

The approximate value of the integrals

$$I_1 = \int_{-1}^{1}\int_{-1}^{1} e^{x+y}\,dxdy = 5.524391382167262$$

$$I_2 = \int_{-1}^{1}\int_{-1}^{1} e^{-\left(x^2+y^2\right)}\,dxdy = 2.230985141404134$$

$$I_3 = \int_{0}^{1}\int_{0}^{1} \frac{\sin^2(x+y)}{(x+y)}\,dxdy = 0.613260369981918$$

Table 1: Approximate solution for various integrals

Integrals	$\left\|I_{S\frac{3}{8}}(f)\right\|$	$\left\|I_{GL2}(f)\right\|$	$\left\|I_{S\frac{3}{8}GL2}(f)\right\|$	$\left\|E_{S\frac{3}{8}GL2}(f)\right\|$
I_1	5.614586191129400	5.488065843621397	5.538673982624600	0.014282600457337
I_2	2.589849108367625	2.024691358024691	2.250754458161865	0.019769316757731
I_3	0.611821774881801	0.609491314229327	0.610423498490317	0.002836871491601

7. Conclusions

Based on the numerical results for three integrals in Table-1, it is clear that the mixed quadrature rule produces better results than the constituent rule for each degree of precision 3. As a result, the mixed quadrature rule is more efficient and numerically closer to the exact result. This manuscript not only evaluates double integrals but also triple integrals, which are frequently used in Mathematical Physics and Applied Sciences for the approximate evaluation of line integrals and surface integrals.

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