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Numerical Solution of Singular Boundary Value Problems by Hermite Wavelet Based Galerkin Method

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Abstract. Singular boundary value problems (SBVPs) occur frequently in various branches of applied mathematics, mechanics, and atomic theory and chemical sciences. In this paper, we proposed the numerical solution of SBVPs by Hermite wavelet based Galerkin method (HWMG). Here, Hermite wavelets are used as weight functions and these are assumed bases elements which allow us to obtain the numerical solutions of the singular boundary value problems. The obtained numerical results using this method are compared with the exact solution and existing methods (FDM, LWGM). Some of the problems are taken to demonstrate the applicability and validity of the proposed method.

Keywords: Hermite wavelet; Galerkin method; Singular boundary value problems

AMS Mathematics Subject Classification (2010): 65T60, 97N40, 30E25

1. Introduction

In recent years, strenuous action and interest have been investigating singular boundary value problems (SBVPs) and a number of methods have been proposed. The singular boundary value problems arise frequently in many branches of applied mathematics, mechanics, and nuclear physics, atomic theory and chemical sciences. Hence, the singular boundary value problems have attracted much attention and have been investigated by many researchers [1,2]. Some of them are Parametric spline method [3], Chebyshev polynomial and B-spline method [4], Laguerre Wavelet based Galerkin Method (LWGM) [5], a new numerical approach [6], etc.

Wavelet analysis is newly developed mathematical tool and have been applied extensively in many engineering filed. This has been received a much interest because of the comprehensive mathematical power and the good application potential of wavelets in science and engineering problems. Special interest has been devoted to the construction of compactly supported smooth wavelet bases. As we have noted earlier that, spectral bases are infinitely differentiable but have global support. On the other side, basis functions used in finite-element methods have small compact support but poor continuity properties. Already we know that, spectral methods have good spectral localization but poor spatial localization, while finite element methods have good spatial localization, but poor spectral localization. Wavelet bases perform to combine the advantages of both spectral and finite element bases. We can expect numerical methods based on wavelet bases to be able to attain good spatial and spectral resolutions [7]. An approach to study differential equations, is the use of wavelet function bases in place of other conventional piecewise polynomial trial functions in finite element type methods. Because of its implementation and simplicity, the Galerkin method is considered the most widely used in applied mathematics [8].

The benefit of wavelet-Galerkin method over finite difference or finite element method has lead to remarkable applications in science and engineering. To a certain extent, the wavelet technique is a strong competitor to the finite element method. Even though the wavelet based method provides an efficient alternative technique for solving singular boundary value problems numerically [5].

In this paper, we developed Hermite wavelet based Galerkin method for the numerical solution of singular boundary value problems. This method is based on expanding the solution by Hermite wavelets with unknown coefficients. The properties of Hermite wavelets together with the Galerkin method are utilized to evaluate the unknown coefficients and then a numerical solution of the singular boundary value problems is obtained.

The organization of the paper is as follows. Preliminaries and properties of Hermite wavelets are given section 2. Section 3 deals with Hermite wavelet based Galerkin method for the solution of the singular boundary value problems. Numerical implementation is given in section 4. Finally, conclusions of the proposed work are discussed in section 5.

2. Preliminaries and properties of Hermite wavelets

Wavelets form a family of functions which are generated from dilation and translation of a single function which is called as mother wavelet $\psi(x)$. If the dialation parameter a and translation parameter b varies continuously, we have the following family of continuous wavelets [9 , 10]:

$$\psi_{a,b}(x) = |a|^{-1/2}\,\psi\left(\frac{x-b}{a}\right), \quad \forall\ a,\ b \in R,\ a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k},\ b = n\,b_0\,a_0^{-k},\ a_0 > 1$,
$b_0 > 0$. We have the following family of discrete wavelets

$$\psi_{k,n}(x) = |a|^{1/2}\,\psi\left(a_0^k\,x - n\,b_0\right), \quad \forall\ a,\ b \in R,\ a \neq 0,$$

where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$,then $\psi_{k,n}$ forms an orthonormal basis. Hermite wavelets are defined as

$$\psi_{n,m}(x) = \begin{cases} \dfrac{2^{\frac{k}{2}}}{\sqrt{\pi}}\tilde{H}_m\,(2^k x - 2n + 1), & \dfrac{n-1}{2^{k-1}} \leq x < \dfrac{n}{2^{k-1}} \\ 0 & ,\qquad \text{otherwise} \end{cases} \tag{2.1}$$

where $$\tilde{H}_m = \sqrt{\frac{2}{\pi}} H_m(x) \tag{2.2}$$

where $m = 0, 1, \ldots, M-1$. In Eq. (2.2) the coefficients are used for orthonormality. Here $H_m(x)$ are the second Hermite polynomials of degree m with respect to weight function $W(x) = \sqrt{1-x^2}$ on the real line R and satisfies the following reccurence formula $H_0(x) = 1$, $H_1(x) = 2x$,

$$H_{m+2}(x) = 2xH_{m+1}(x) - 2(m+1)H_m(x), \text{ where } m = 0, 1, \ldots . \tag{2.3}$$

For $k = 1$ and $n = 1$ in Eq. (2.1) and (2.2), then the Hermite wavelets are given by

$$\psi_{1,0}(x) = \frac{2}{\sqrt{\pi}},$$

$$\psi_{1,1}(x) = \frac{2}{\sqrt{\pi}}(4x-2),$$

$$\psi_{1,2}(x) = \frac{2}{\sqrt{\pi}}(16x^2 - 16x + 2),$$

$$\psi_{1,3}(x) = \frac{2}{\sqrt{\pi}}(64x^3 - 96x^2 + 36x - 2),$$

$$\psi_{1,4}(x) = \frac{2}{\sqrt{\pi}}(256x^4 - 512x^3 + 320x^2 - 64x + 2)$$, and so on.

Function approximation:

We would like to bring a solution function $y(x)$ under Hermite space by approximating $y(x)$ by elements of Hermite wavelet bases as follows,

$$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\, \psi_{n,m}(x) \tag{2.4}$$

where $\psi_{n,m}(x)$ is given in Eq. (2.1).

We approximate $y(x)$ by truncating the series represented in Eq. (2.4) as,

$$y(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) \tag{2.5}$$

where $c_{n,m}$ and ψ are $2^{k-1} M \times 1$ matrix.

Convergence of Hermite wavelets

Theorem 2.1. If a continuous function $y(x) \in L^2(R)$ defined on $[0, 1)$ be bounded, i.e. $y(x) \leq K$, then the Hermite wavelets expansion of $y(x)$ converges uniformly to it [11].

3. Method of solution

Consider the singular boundary value problem is of the form,

$$\frac{d^2y}{dx^2}+\frac{\alpha}{x}\frac{dy}{dx}+p(x)y=q(x) \tag{3.1}$$

With boundary conditions $y(0)=a, y(1)=b$ (3.2)

where $p(x)$ and $q(x)$ are analytic in $x \in [0,1)$ and a, b and α are finite constants. In Eq. (3.1) has singularity at the initial point $x = 0$. We note that the main difficulty arises in the singularity of the equation at $x = 0$.

Write the Eq. (3.1) as

$$R(x)=\frac{d^2y}{dx^2}+\frac{\alpha}{x}\frac{dy}{dx}+p(x)y-q(x) \tag{3.3}$$

where $R(x)$ is the residual of the Eq. (3.1). When $R(x) = 0$ for the exact solution, $y(x)$ only which will satisfy the boundary conditions.

Consider the trail series solution of the Eq. (3.1), $y(x)$ defined over $[0,1)$ can be expanded as a modified Hermite wavelet, satisfying the given boundary conditions which is involving unknown parameter as follows,

$$y(x) = \sum_{i=1}^{2^{k-1}} \sum_{j=1}^{M} c_{i,j}\,\psi_{i,j}(x) \tag{3.4}$$

where $c_{i,j}$'s are unknown coefficients to be determined.

Accuracy in the solution is increased by choosing higher degree Hermite wavelet polynomials.

Differentiating Eq. (3.4) twice with respect to x and substitute the values of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$, y in Eq. (3.3). To find $c_{i,j}$'s we choose weight functions as assumed bases elements and integrate on boundary values together with the residual to zero [12].

i.e. $$\int_0^1 \psi_{1,j}(x)\,R(x)\,dx = 0, \quad j = 0,1,2,\ldots\ldots,n$$

then we obtain a system of linear equations, on solving this system, we get unknown parameters. Then substitute these unknowns in the trail solution, numerical solution of Eq. (3.1) is obtained.

4. Numerical implementation

In this section, we applied Hermite wavelet based Galerkin method for the numerical solution singular boundary value problems and subsequently presented the efficiency of the method in the form of tables and figures. The error analysis is considered as

i.e. Absolute error $= E_{max} = \max\left|y_e - y_a\right|$,

where y_e and y_a are exact and approximate solutions respectively.

Problem 4.1 Consider the singular boundary value problem [13],

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}-\frac{2}{x^2}y = 4, \quad 0 \le x \le 1 \tag{4.1}$$

With boundary conditions: $y(0) = 0$, $y(1) = 0$ (4.2)

The implementation of the Eq. (4.1) as per the method explained in section 3 is as follows:
Now rewrite the Eq. (4.1) as

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-2y=4x^2$$

and its residual can be written as:

$$R(x)=x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-2y-4x^2 \tag{4.3}$$

Now, choosing the weight function $w(x) = x(1-x)$ for Hermite wavelet bases $(\text{for } k = 1 \text{ and } m = 3)$ to satisfy the given boundary conditions Eq. (4.2), i.e.

$\psi(x) = w(x)\times\psi(x)$

$$\psi_{1,0}(x) = \psi_{1,0}(x)\times x(1-x) = \frac{2}{\sqrt{\pi}}x(1-x) \quad ,$$

$$\psi_{1,1}(x) = \psi_{1,1}(x)\times x(1-x) = \frac{2}{\sqrt{\pi}}(4x-2)x(1-x)$$

$$\psi_{1,2}(x) = \psi_{1,2}(x)\times x(1-x) = \frac{2}{\sqrt{\pi}}(16x^2-16x+2)x(1-x)$$

Assuming the trail solution of (4.1) for $k=1$ and $m=3$ is given by

$$y(x) = c_{1,0}\,\psi_{1,0}(x) + c_{1,1}\,\psi_{1,1}(x) + c_{1,2}\,\psi_{1,2}(x) \tag{4.4}$$

Then the Eq. (4.4) becomes

$$\begin{aligned} y(x) = c_{1,0}\frac{2}{\sqrt{\pi}}x(1-x) + c_{1,1}\frac{2}{\sqrt{\pi}}(4x-2)x(1-x) + \\ c_{1,2}\frac{2}{\sqrt{\pi}}(16x^2-16x+2)x(1-x) \end{aligned} \tag{4.5}$$

Differentiating Eq. (4.5) twice w.r.t. x, substitute the values of y , $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in Eq. (4.3), we get the residual of Eq. (4.1).

The "weight functions" are the same as the bases functions. Then by the weighted Galerkin method, we consider the following:

$$\int_0^1 \psi_{1,j}(x)\,R(x)dx = 0, \quad j = 0,1,2 \tag{4.6}$$

For $j = 0,1,2$ in Eq. (4.6),

i.e. $\int_0^1 \psi_{1,0}(x)\,R(x)dx = 0$, $\int_0^1 \psi_{1,1}(x)\,R(x)dx = 0$, $\int_0^1 \psi_{1,2}(x)\,R(x)dx = 0$ (4.7)

From Eq. (4.7), we have system of algebraic equations with unknown coefficients i.e. $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$. Solving this by Gauss elimination method, we obtain the values of $c_{1,0}=-0.8945$, $c_{1,1}=0.0047$ and $c_{1,2}=-0.0046$. Substituting these values in Eq.

(4.5), we get the numerical solution; the absolute errors are presented in table 1 and comparison with exact solution of Eq. (4.1) is $y(x) = x^2 - x$ in figure 1.

Table 1. Comparison of numerical solution with exact solution and absolute errors of the Problem 4.1

X	Numerical solution			Exact solution	Absolute error		
	FDM	LWGM [5]	HWGM		FDM	LWGM [5]	HWGM
0.1	-0.01121	-0.08668	-0.091865	-0.09000	7.88e-02	3.32e-03	1.90e-03
0.2	-0.02727	-0.15682	-0.162047	-0.16000	1.33e-02	3.18e-03	2.00e-03
0.3	-0.04425	-0.20842	-0.211369	-0.21000	1.66e-02	1.58e-03	1.40e-03
0.4	-0.06055	-0.24013	-0.240457	-0.24000	1.79e-02	1.30e-04	4.60e-04
0.5	-0.07470	-0.25119	-0.249739	-0.25000	1.75e-02	1.19e-03	2.60e-04
0.6	-0.08470	-0.24133	-0.239439	-0.24000	1.55e-02	1.33e-03	5.60e-04
0.7	-0.08765	-0.21070	-0.209587	-0.21000	1.22e-02	7.00e-04	4.10e-04
0.8	-0.07921	-0.15977	-0.160010	-0.16000	8.08e-02	2.30e-04	1.00e-05
0.9	-0.05306	-0.08924	-0.090338	-0.09000	3.69e-02	7.60e-04	3.40e-04

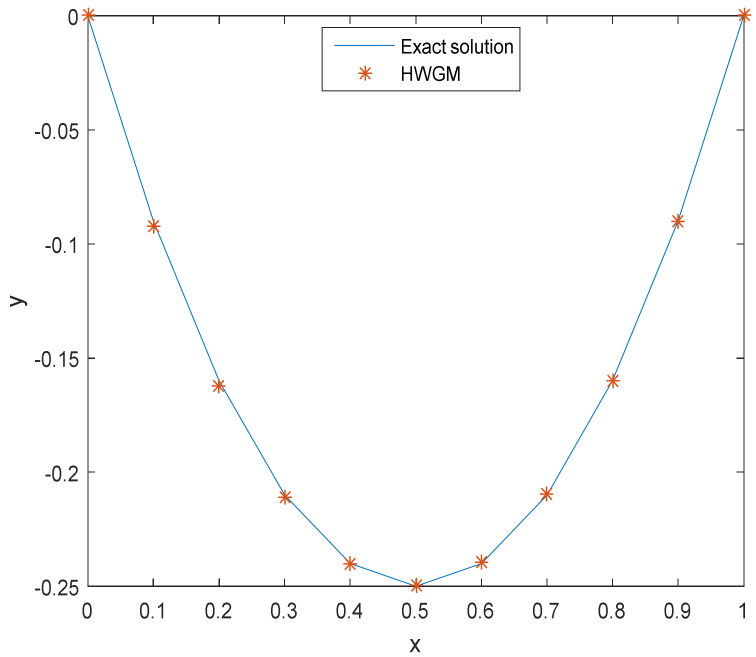

Figure 1: Comparison of numerical solution and exact solution of the problem 4.1

Problem 4.2 Consider, another singular boundary value problem [14]

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = x^2 - x^3 - 9x + 4, \quad 0 \le x \le 1 \tag{4.8}$$

With boundary conditions: $y(0) = 0, \quad y(1) = 0$ (4.9)

By applying the method explained in the section 3 and as in the previous example, we obtained the we obtain the values of $c_{1,0} = 0.4429$, $c_{1,1} = 0.2217$ and $c_{1,2} = -0.0001$. Substituting these values in Eq. (4.5), we get the numerical solution; the absolute errors are presented in table 2 and comparison with exact solution of Eq. (4.8) is $y(x) = x^2 - x^3$ in figure 2.

Table 2. Comparison of numerical solution with exact solution and absolute errors of the Problem 4.2

x	Numerical solution			Exact solution	Absolute error		
	FDM	LWGM [5]	HWGM		FDM	LWGM [5]	HWGM
0.1	-0.014709	0.010673	0.008949	0.009000	2.37e-02	1.67e-03	5.10e-05
0.2	-0.013726	0.033159	0.031941	0.032000	4.57e-02	1.16e-03	5.90e-05
0.3	-0.002584	0.063290	0.062954	0.063000	6.56e-02	2.90e-04	4.60e-05
0.4	0.015387	0.095881	0.095977	0.096000	8.06e-02	1.19e-04	2.30e-05
0.5	0.036564	0.125034	0.124996	0.125000	8.84e-02	3.40e-05	4.00e-06
0.6	0.056572	0.144429	0.144008	0.144000	8.74e-02	4.29e-04	8.00e-06
0.7	0.070066	0.147623	0.147009	0.147000	7.69e-02	6.23e-04	9.00e-06
0.8	0.070568	0.128350	0.128003	0.128000	5.74e-02	3.50e-04	3.00e-06
0.9	0.050294	0.080816	0.080996	0.081000	3.07e-02	1.84e-04	4.-00e-06

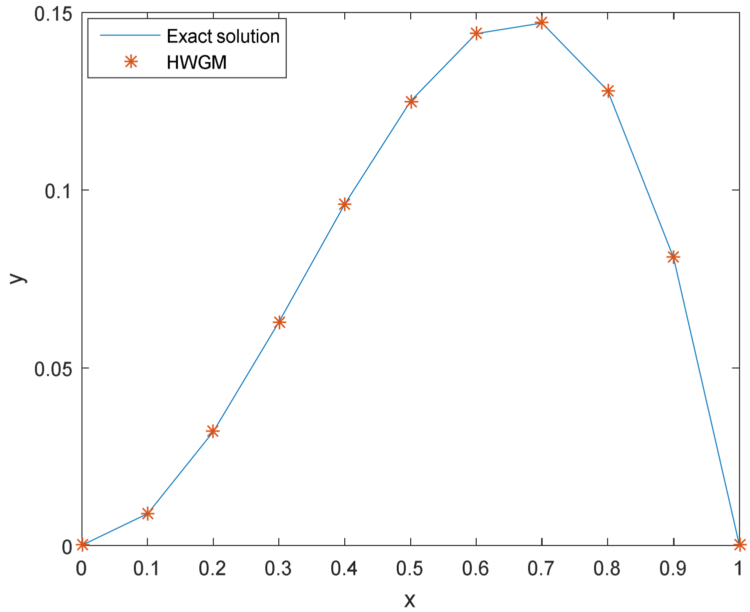

Figure 2. Comparison of numerical and exact solution of the problem 4.2.

Problem 4.3 Finally, consider singular boundary value problem [15]

$$\frac{d^2 y}{d x^2} + \frac{8}{x}\frac{d y}{d x} + x y = x^5 - x^4 + 44 x^2 - 30 x, \quad 0 \le x \le 1 \quad (4.10)$$

With boundary conditions: $y(0) = 0$, $y(1) = 0$ (4.11)

By applying the method explained in the section 3 and as in the previous examples, we obtained the values of $c_{1,0} = -0.3324$, $c_{1,1} = -0.2216$ and $c_{1,2} = -0.0554$. Substituting these values in Eq. (4.5), we get the numerical solution; the absolute errors are presented in table 3 and comparison with exact solution of Eq. (4.10) is $y(x) = -x^3 + x^4$ in figure 3.

Table 3. Comparison of numerical solution with exact solution and absolute errors of the problem 4.3

x	Numerical solution			Exact solution	Absolute error		
	FDM	LWGM [5]	HWGM		FDM	LWGM [5]	HWGM
0.1	0.024647	-0.000823	-0.000900	-0.000900	2.55e-02	7.70e-05	0
0.2	0.024538	-0.004844	-0.006401	-0.006400	3.09e-02	1.56e-03	1.00e-06
0.3	0.016024	-0.016861	-0.018904	-0.018900	3.40e-02	2.04e-03	4.00e-06
0.4	-0.000072	-0.037304	-0.038407	-0.038400	3.83e-02	1.10e-03	7.00e-06
0.5	-0.022021	-0.062986	-0.062512	-0.062500	4.05e-02	4.86e-04	1.20e-05
0.6	-0.045926	-0.087854	-0.086417	-0.086400	4.05e-02	1.45e-03	1.70e-05
0.7	-0.065532	-0.103744	-0.102920	-0.102900	3.74e-02	8.44e-04	2.00e-05
0.8	-0.072190	-0.101131	-0.102420	-0.102400	3.02e-02	1.27e-03	2.00e-05
0.9	-0.054840	-0.069880	-0.072914	-0.072900	1.81e-02	3.02e-03	1.40e-05

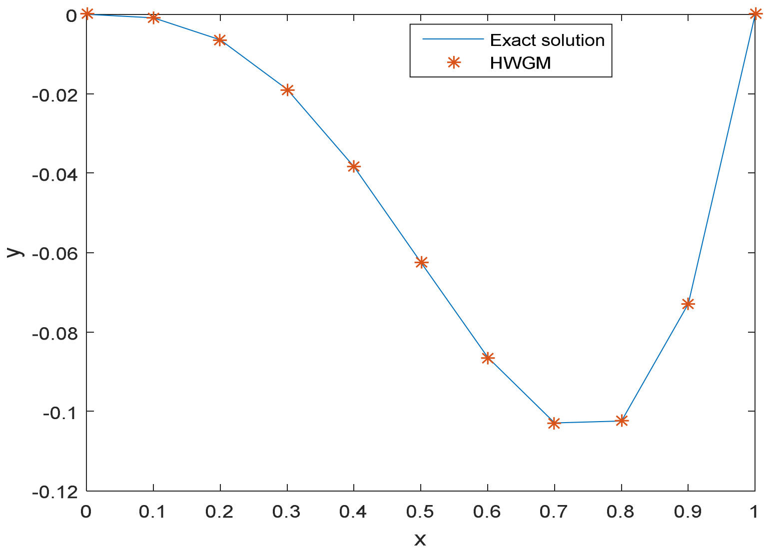

Figure 3: Comparison of numerical solution and exact solution of the problem 4.3.

5. Conclusion

In this paper, we proposed the Hermite wavelet based Galerkin method for the numerical solution of singular boundary value problems (SBVPs). From the above tables and figures, we observed that the numerical solutions obtained by the proposed method are better than FDM, Laguerre wavelet based Galerkin method (LWGM) and nearer to the exact solution. Hence, the Laguerre wavelet based Galerkin method (LWGM) is effective for solving singular boundary value problems.

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