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On the Exponential Diophantine Equation $(p+2)^x + (5p+6)^y = z^2$ when $p, p+2$ and $5p+6$ are Primes

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Abstract. In this work, we prove that the exponential Diophantine equation $(p+2)^x + (5p+6)^y = z^2$ has no solution when $p, p+2$ and $5p+6$ are primes, and x, y, z are non-negative integers.

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1. Introduction

Over a decade, Exponential Diophantine equation has been a famous topic in number theory. After, Catalan presented Catalan's conjecture that is $a^x - b^y = 1$ and Mihailescu proved its solution [4, 12]. There are many articles using Catalan's conjecture to find and prove solution of the equation in general form

$$p^x + q^y = z^2,$$

where p, q, x, y and z are non-negative integers [8, 10-11, 13-15]. In 2013, the equation $p^x + (p+1)^y = z^2$ where x, y and z are non-negative integers and p is Mersenne prime was presented and proved the solutions [5]. In the proof, Catalan's conjecture was applied. The result indicates that there are two solutions including $(p, x, y, z) = (7, 0, 1, 3)$ and $(p, x, y, z) = (3, 2, 2, 5)$. In 2018, Burshtein presented both the Exponential Diophantine equations $p^x + (p+4)^y = z^2$ when $p > 3, p+4$ are prime and $p^x + (p+6)^y = z^2$ when $p, p+6$ are primes [1, 2]. He proved that the first equation has no solution and the other has seven solutions for $x + y = 2, 3, 4$. In the same year, the equation $p^x + (p+8)^y = z^2$ when $p > 3$ and $p+8$ are primes was studied by

Oliveira [7]. He proved that the equation has no solution. Next year, a research group proved that $p^x+(p+12)^y=z^2$ has no solution [9]. In 2020, Burshtein [3] proved that the Diophantine equation $p^x+(p+5)^y=z^2$ has no solution when $p+5=2^{2u}$ and u is a positive integer. Recently, the Diophantine equation $p^x+(p+20)^y=z^2$ when p and $p+20$ are primes was proved that it has no solution [6].

In this paper, we studied the Exponential Diophantine equation $(p+2)^x+(5p+6)^y=z^2$ when $p, p+2$ and $5p+6$ are primes, and x,y,z are non-negative integers.

2. Preliminaries

In this section, we introduce basic knowledge applying in the proof.

Lemma 2.1. If x is odd then $x\equiv 1(\mathrm{mod}\,4)$ or $x\equiv -1(\mathrm{mod}\,4)$.

Proof: Let x be odd. There exists $q\in$ where $x=4q+1$ or $x=4q+3$. If $x=4q+1$ then we have $4\,|\,x-1$ or $x\equiv 1(\mathrm{mod}\,4)$. If $x=4q+3$ then we have $x=4(q+1)-1$. This yields $4\,|\,x+1$ or $x\equiv -1(\mathrm{mod}\,4)$. □

Lemma 2.2. If x is even then $x^2\equiv 0(\mathrm{mod}\,4)$.

Proof: Let x be even. We can write $x=2m$ where $m\in\mathbb{Z}^+$. Then, we obtain $x^2=4m^2$ which implies that $4\,|\,x^2$ or $x^2\equiv 0(\mathrm{mod}\,4)$.□

Lemma 2.3. (Catalan's conjecture) [12] The Diophantine equation $a^x-b^y=1$ has only one solution that is $(a,b,x,y)=(3,2,2,3)$ where a,b,x and y are integers and $\min\{a,b,x,y\}>1$.

Lemma 2.4. The Diophantine equation $1+p^x=z^2$ where $p\geq 5$ is prime and x,z are non-negative integers has no solution.

Proof: Let x and z be non-negative integers such that

$$1+p^x=z^2 \tag{1}$$

where p is prime and $p\geq 5$. We consider in 3 cases including $x=0$, $x=1$ and $x\geq 2$.

For $x=0$, (1) becomes $z^2=2$ which is impossible. For $x=1$, (1) becomes $1+p=z^2$. This implies that $z^2\geq 6$ or $z\geq 3$, and we also obtain $p=(z-1)(z+1)$ which is impossible. For $x\geq 2$, we have $z^2-p^x=1$. By Lemma 2.3, the equation has only one solution that is $(z,p,x)=(3,2,3)$. This is impossible because of $p\geq 5$. Therefore, (1) has no solution. □

3. Main results

Theorem 3.1. If $p, p+2$ and $5p+6$ are primes then the Exponential Diophantine equation $(p+2)^x+(5p+6)^y=z^2$ has no solution in non-negative integers. x, y and z.

Proof: Let x, y and z are non-negative integers according to

$$(p+2)^x+(5p+6)^y=z^2. \quad (2)$$

We consider in 3 cases as following.

Case 1: $x=0$, (2) becomes $1+(5p+6)^y=z^2$. Since $5p+6$ is prime and $5p+6\geq 5$. By Lemma 2.4, the equation has no solution.

Case 2: $y=0$, (2) becomes $1+(p+2)^x=z^2$. Because $p+2$ is prime, it implies that $p+2\geq 5$. By Lemma 2.4, the equation has no solution.

Case 3: $x>0$ and $y>0$, since p and $p+2$ are primes, we obtain $p\neq 2$ and p is odd. Therefore, we consider in two cases including $p\equiv 1 \pmod 4$ and $p\equiv -1 \pmod 4$.

For $p\equiv 1 \pmod 4$, we can see that (2) becomes

$$z^2=(-1)^x+(-1)^y \pmod 4. \quad (3)$$

From (1), z is even and we obtain $z^2\equiv 0 \pmod 4$ by Lemma 2.2. From (3), x and y are consider in two cases. One is " x is even and y is odd". The other is " x is odd and y is even".

For " x is even and y is odd", we have $x=2k$ where $k\in\mathbb{Z}^+$. From (2), we obtain

$$(5p+6)^y=z^2-(p+2)^{2k}=\left[z-(p+2)^k\right]\left[z+(p+2)^k\right].$$

Hence, there are $\alpha,\beta\in$ such that $z-(p+2)^k=(5p+6)^\alpha$ and $z+(p+2)^k=(5p+6)^\beta$ where $0\leq\alpha<\beta$ and $\alpha+\beta=y$.

Then, we have

$$2(p+2)^k=(5p+6)^\beta-(5p+6)^\alpha=(5p+6)^\alpha\left((5p+6)^{\beta-\alpha}-1\right).$$

It easy to check that $\alpha=0$. So we have

$$2(p+2)^k=(5p+6)^\beta-1=5(p+1)\left((5p+6)^{\beta-1}+(5p+6)^{\beta-2}+\cdots+(5p+6)+1\right).$$

Thus, we have $5\mid p+2$. This implies that $p+2=5$, then $p=3$. We obtain that $5p+6=5(3)+6=21$ which is not a prime. This is a contradiction.

For " x is odd and y is even", let $y=2k$, where $k\in\mathbb{Z}^+$. From (2), we obtain

$$(p+2)^x=z^2-(5p+6)^{2k}=\left(z-(5p+6)^k\right)\left(z+(5p+6)^k\right).$$

We let $\alpha, \beta \in \mathbb{N}$ where $0 \le \alpha < \beta$ and $\alpha + \beta = x$. We have $z-(5p+6)^k = (p+2)^\alpha$ and $z+(5p+6)^k = (p+2)^\beta$. It's easy to see that

$$2(5p+6)^k = (p+2)^\alpha\left((p+2)^{\beta-\alpha}-1\right).$$

It implies that $\alpha = 0$, then we obtain

$$2(5p+6)^k = (p+2)^\beta - 1 = (p+1)\left((p+2)^{\beta-1}+(p+2)^{\beta-2}+\cdots+1\right). \quad (4)$$

There exist $m \in \mathbb{Z}^+ - \{1\}$ such that $p+1=2m$, since $p+1 \ge 4$ and $p+1$ are even. From (4), we obtain

$$(5p+6)^k = m\left((p+2)^{\beta-1}+(p+2)^{\beta-2}+\cdots+1\right). \quad (5)$$

There exist a prime q where is $q \mid m$. From (5), we obtain $q \mid (5p+6)^k$. We can see that $q \mid (5p+6)$, but $5p+6$ is prime. Hence, we have $q = 5p+6$. This is contradiction because $q \le p+1 < 5p+6$.

For $p \equiv -1 (\mathrm{mod}\, 4)$, we obtain $p+2 \equiv 1(\mathrm{mod}\, 4)$ and $5p+6 \equiv 1(\mathrm{mod}\, 4)$. From (2), we have

$$z^2 \equiv 2(\mathrm{mod}\, 4).$$

This is contradiction because $z^2 \equiv 0,1(\mathrm{mod}\, 4)$. This completes the proof of theorem 3.1. □

4. Conclusion

In this work, we prove that the Exponential Diophantine equation $(p+2)^x + (5p+6)^y = z^2$ when $p, p+2$ and $5p+6$ are primes, and x, y, z are non-negative integers. In the proof, we separate in 3 cases including case $x=0$, case $y=0$ and case $x>0$ and $y>0$. The Catalan's conjecture and principles in number theory are applied. The results reveals that the equation has no solution.

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