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On the Non-Linear Diophantine Equation $p^x + (p + 4^n)^y = z^2$ where p and $p + 4^n$ are Primes

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Abstract. In this paper, we consider the non-linear Diophantine equation $p^x + (p + 4^n)^y = z^2$, where $p > 3, p + 4^n$ are primes, *x*, *y* and *z* are nonnegative integers and n is a natural number. It is shown that this non-linear Diophantine equation has no solution.

Keywords: Diophantine equations, exponential equations

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

Many studies claim that the Diophantine equation is one of the classic problems in elementary number theory and algebraic number theory. In 1844, Catalan [6] proved that a conjecture (a, b, x, y) = (3, 2, 2, 3) is a unique solution of the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with min{a, b, x, y} > 1.

Later in 2014, Suvarnamani [12] proved that the equation $p^x + (p + 1)^y = z^2$ is the unique solution (p, x, y, z) = (3, 1, 0, 2) when p is an odd prime and x, y, z are non-negative integers.

In 2017, Burshtein [2] examined the Diophantine equation $p^3 + q^2 = z^4$ when p is Prime has no solution in positive integers.

In 2018, Burshtein [3] studied solutions to the Diophantine equation $M^x + (M + 6)^y = z^2$ when M = 6N + 5 and M, M + 6 are primes has no solutions.

Additionally in 2018, Kumar et al. [9,10] showed that on the non-linear Diophantine equation $p^x + (p + 6)^y = z^2$, when p and p+6 both are primes with p = 6n+1 has no solution, where x, y, and z are non-negative integer and n is a natural number on the non-linear Diophantine equation, $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$.

Moreover, Fernando [8] also showed that the Diophantine equation $p^x + (p + 8)^y = z^2$ when p > 3 and p + 8 are primes has no solution (x, y, z) in positive integers.

Kumar et al. [11] proved that on the solution of exponential Diophantine equation $p^x + (p + 12)^y = z^2$ where x, y, z are non-negative integers and p, 2 are primes such that p is of the form 6n + 1, where n is a natural number. They proved that this Exponential Diophantine equation has no non-negative integral solution.

In 2020, Burshtein [4,5] showed that the Diophantine equation $p^x + (p+5)^y = z^2$ when p is prime where $p+5 = 2^{2u}$ has no solution (x, y, z) in positive integers and proved that on solution to the Diophantine equation $p^x + q^y = z^3$ when $p \geq 2, q$ are primes. $1 \leq x, y \leq 2$ are integers.

Dokchan and Pakapongpun [7] put that on the Diophantine equation $p^x + (p+20)^y = z^2$ where p and p + 20 are primes which has been proved that it has no solution.

Because of this open problem, the author is therefore interested in study the Diophantine equation; $p^x + (p+4^n)^y = z^2$ has no solutions. , where x, y, z are non-negative integers and p >3 and $p+4^n$ are primes and n is natural number.

2. Preliminaries

Lemma 2.1. The Diophantine equation $1 + (p+4^n)^y = z^2$ has no solutions where p > 3, $p+4^n$ are primes and n is natural number y and z are non-negative integers.

Proof: Since $1 + (p+4^n)^y = z^2$, z is even and so $z^2 \equiv 0 (mod\ 4)$. Since $p > 3$ and $p+4^n$ are primes, $p \equiv 1(mod\ 4)\ or\ p \equiv -1 \equiv 3\ (mod\ 4)$.

Case I: For $p \equiv 1(mod\ 4)$, $1 + (p+4^n)^y \equiv 2(mod\ 4)$ which is a contradiction since $z^2 \equiv 0(mod\ 4)$.

Case II: Suppose p ≡ −1 (mod 4).

If y = 2s, s ≥ 1, then $1 + (p+4^n)^y \equiv 2(mod\ 4)$. which is a contradiction since $z^2 \equiv 0(mod\ 4)$.

If y = 2s +1, s ≥ 0, then $1 + (p+4^n)^y = z^2$or equivalently $(p+4^n)^{2s+1} = (z-1)(z+1)$. Thus there exist non-negative integers α, β such that $(p+4^n)^\alpha = z+1$and $(p+4^n)^\beta = z-1$, where $\alpha > \beta$ and α+β = 2s+1. Therefore $(p+4^n)^\beta\left((p+4^n)^{\alpha-\beta} - 1\right) = 2$ This implies that $\beta = 0$ and $(p+4^n)^{2s+1} - 1 = 2$. Then $(p+4^n)^{2s+1} = 3$. which is impossible.

Hence the Diophantine equation $1 + (p+4^n)^y = z^2$ has no solutions where p > 3, $p+4^n$ are primes and n is natural number y and z are non-negative integers.

Lemma 2.2. The Diophantine equation $p^x + 1 = z^2$ has no solutions where p > 3, $p+4^n$ are primes and n is natural number x and z are non-negative integers.

Proof: Since $p^x + 1 = z^2$, z is even and so $z^2 \equiv 0$ (mod 4).

Since p > 3, p is prime, p ≡ 1 (mod 4) or p ≡ −1 ≡ 3 (mod 4).

Case I: Suppose p ≡ 1 (mod 4).

then $p^x + 1 \equiv 2$ (mod 4) which is a contradiction since $z^2 \equiv 0$ (mod 4).

Case II: Suppose p ≡ −1 (mod 4).

If x = 2k, k ≥ 1 , then $p^x + 1 \equiv 2$ (mod 4). which is a contradiction since $z^2 \equiv 0$ (mod 4).

If x = 2k +1, k ≥ 0, then $p^{2k+1} + 1 = z^2 = (z+1)(z-1)$. Thus there exist non-negative integers α, β such that $p^\alpha = z+1$ and $p^\beta = z-1$, where α > β and α+β = 2k+1. Therefore, $p^\beta (p^{\alpha-\beta} - 1) = 2$. This implies that β = 0 and $p^{2k+1} - 1 = 2$. Then $p^{2k+1} = 3$ which is impossible.

Hence the Diophantine equation $p^x + 1 = z^2$ has no solutions where p > 3, $p+4^n$ are primes and n is natural number y and z are non-negative integers.

3. Main theorem

Theorem 3.1. The Diophantine equation $p^x + (p+4^n)^y = z^2$, where p > 3, $p+4^n$ are primes and n is natural number y and z are non-negative integers.

Proof: Since $p^x + (p+4^n)^y = z^2$, z is even and so $z^2 \equiv 0 (mod\ 4)$.
Since p > 3, $p+4^n$ are primes, $p \equiv 1 \pmod 4$ or $p \equiv -1 \equiv 3 \pmod 4$.
Then we consider in 4 cases as follows;

Case 1. If x = 0, y= 0, then $z^2 = 2$ which is impossible.

Case 2. If x = 0 , $y \geq 1$, then $1 + (p+4^n)^y = z^2$. which has no solution by Lemma 2.1.

Case 3. If y = 0 , $x \geq 1$, then $p^x + 1 = z^2$ which has no solution by Lemma 2.2.

Case 4. If $x \geq 1$, $y \geq 1$,then $p^x + (p+4^n)^y = z^2$ has no solutions.

We consider in 4 subcases as follow;

Subcase 1: If x = 2k, $k \geq 1$ and $y = 2s, s \geq 1$,then
$p^x + (p+4^n)^y \equiv 2 (mod\ 4)$. which is a contradiction since $z^2 \equiv 0 (mod\ 4)$.

Subcase 2: If x = 2k+1, $k \geq 0$, $y = 2s+1, s \geq 0$, then
$p^x + (p+4^n)^y \equiv 2\ (mod\ 4)$. which is a contradiction since $z^2 \equiv 0 (mod\ 4)$.

Subcase 3: If x = 2k+1, $k \geq 0$, $y = 2s, s \geq 1$, then $p^x + (p+4^n)^y = z^2$. Thus there exist non-negative integers α, β such that
$p^\alpha = z + (p+4^n)^s$ and $p^\beta = z - (p+4^n)^s$, where $\alpha > \beta$ and α+β = 2k+1. Therefore $p^\beta(p^{\alpha-\beta} - 1) = 2(p+4^n)^s$ This implies that $\beta = 0$ and $p^{2k+1} - 1 = 2(p+4^n)^s$.

For k = 0, we obtain $p - 1 = 2(p+4^n)^s$. Or $p = 2(p+4^n)^s + 1$, which is impossible.

For $k \geq 1$, we have $p^{2k+1} - 1 = 2(p+4^n)^s = (p-1)(p^{2k} + p^{2k-1} + \cdots + p + 1)$. It follows that p − 1 is an even positive divisor of $2(p+4^n)^s$, that is $p-1 = 2(p+4^n)^j$ where j is an integer such that $0 \leq j < s$. For j = 0, p = 3 which contradicts the fact that p > 3. For $1 \leq j < s$, we obtain $2(p+4^n)^j = (p+4) - 5$ or $2(p+4^n)^j + 5 = p + 4$ which is impossible. Therefore, the Diophantine equation $p^x + (p+4^n)^y = z^2$, where p > 3 and $p+4^n$ are primes, has no solutions.

Subcase 4: If x = 2k, $k \geq 1$, $y = 2s+1, s \geq 0$, then $p^x + (p+4^n)^y = z^2$. Thus there exist non-negative integers α, β such that $(p+4^n)^\alpha = z + p^k$ and
$(p+4^n)^\beta = z - p^k$, where $\alpha > \beta$ and α+β = 2s+1. Therefore $(p+4^n)^\beta((p+4^n)^{\alpha-\beta} - 1) = 2p^k$ This implies that $\beta = 0$ and $(p+4^n)^\alpha - 1 = 2p^k$, $(p+4^n)^{2s+1} - 1 = 2p^k$. For s = 0, $(p+4^n) - 1 = 2p^k$. Or $(2p^{k-1} - 1) = 4^n - 1 = 3\ (4^{n-1} + 4^{n-2} + \cdots + 1)$. which is impossible.

For $s \geq 1$, we have $(p+4^n)^{2s+1} - 1 = 2p^k$, $2p^k = (p + 4^n - 1)((p+4^n)^{2s} + (p+4^n)^{2s-1} + \cdots + (p+4^n) + 1)$.
It follows that $p + 4^n - 1$ is an even positive divisor of $2p^k$, that is

$p+4^n-1 = 2(p+4^n)^j$. where j is an integer such that 0 ≤ j < s. For j = 0, $p+4^n$ = 3 which is impossible. For 1 ≤ j < s, we obtain $p+4^n-1 = 2(p+4^n)^j$. Or $p+4^n = 2(p+4^n)^j+1$which is impossible. Therefore, the Diophantine equation $p^x+(p+4^n)^y=z^2$. has no solutions, where p > 3 and $p+4^n$ are primes , n is natural number, x, y and z are non − negative integers.

Corollary 3.1.1. The Diophantine equation $p^x+(p+4^n)^y=u^{2n}$ has no solution., where u,x, y and z are non-negative integers and n is a natural number.
Proof: Let $u^n=z$ then $p^x+(p+4^n)^y=u^{2n}=z^2$, which has no solution by Theorem 3.1.

Corollary 3.1.2. The Diophantine equation $p^x+(p+4^n)^y=u^{2n+2}$ has no solution, where u,x, y and z are non-negative integers and n is a natural number.
Proof: Let $u^{n+1}=z$ then $p^x+(p+4^n)^y=u^{2n+2} = z^2$, which has no solution by Theorem 3.1.

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