On the Non-Linear Diophantine Equation

\[ p^x + (p + 4^n)^y = z^2 \] where \( p \) and \( p + 4^n \) are Primes

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Abstract. In this paper, we consider the non-linear Diophantine equation \( p^x + (p + 4^n)^y = z^2 \), where \( p > 3, p + 4^n \) are primes, \( x, y \) and \( z \) are nonnegative integers and \( n \) is a natural number. It is shown that this non-linear Diophantine equation has no solution.

Keywords: Diophantine equations, exponential equations

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1. Introduction
Many studies claim that the Diophantine equation is one of the classic problems in elementary number theory and algebraic number theory. In 1844, Catalan [6] proved that a conjecture \((a, b, x, y) = (3, 2, 2, 3)\) is a unique solution of the Diophantine equation \( a^x - b^y = 1 \) where \( a, b, x \) and \( y \) are integers with \( \min\{a, b, x, y\} > 1 \).

Later in 2014, Suvarnamani [12] proved that the equation \( p^x + (p + 1)^y = z^2 \) is the unique solution \((p, x, y, z) = (3, 1, 0, 2)\) when \( p \) is an odd prime and \( x, y, z \) are non-negative integers.

In 2017, Burshtein [2] examined the Diophantine equation \( p^3 + q^2 = z^4 \) when \( p \) is Prime has no solution in positive integers.

In 2018, Burshtein [3] studied solutions to the Diophantine equation \( M^x + (M + 6)^y = z^2 \) when \( M = 6N + 5 \) and \( M, M + 6 \) are primes has no solutions.

Additionally in 2018, Kumar et al. [9,10] showed that on the non-linear Diophantine equation \( p^x + (p + 6)^y = z^2 \), when \( p \) and \( p+6 \) both are primes with \( p = 6n+1 \) has no solution, where \( x, y, \) and \( z \) are non-negative integer and \( n \) is a natural number on the non-linear Diophantine equation, \( 61^x + 67^y = z^2 \) and \( 67^x + 73^y = z^2 \).

Moreover, Fernando [8] also showed that the Diophantine equation \( p^x + (p + 8)^y = z^2 \) when \( p > 3 \) and \( p + 8 \) are primes has no solution \((x, y, z)\) in positive integers.

Kumar et al. [11] proved that on the solution of exponential Diophantine equation \( p^x + (p + 12)^y = z^2 \) where \( x, y, z \) are non-negative integers and \( p, 2 \) are primes such that \( p \) is of the form \( 6n + 1 \), where \( n \) is a natural number. They proved that this Exponential Diophantine equation has no non-negative integral solution.
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In 2020, Burshtein [4,5] showed that the Diophantine equation
\( p^x + (p + 5)^y = z^2 \) when \( p \) is prime where \( p + 5 = 2^{2u} \) has no solution \((x, y, z)\) in positive integers and proved that on solution to the Diophantine equation \( p^x + q^y = z^3 \) when \( p \geq 2, q \) are primes. \( 1 \leq x, y \leq 2 \) are integers.

Dokchan and Pakapongpun [7] put that on the Diophantine equation
\( p^x + (p + 20)^y = z^2 \) where \( p \) and \( p + 20 \) are primes which has been proved that it has no solution.

Because of this open problem, the author is therefore interested in study the Diophantine equation; \( p^x + (p + 4^n)^y = z^2 \) has no solutions., where \( x, y, z \) are non-negative integers and \( p > 3 \) and \( p + 4^n \) are primes and \( n \) is natural number.

2. Preliminaries

Lemma 2.1. The Diophantine equation \( 1 + (p + 4^n)^y = z^2 \) has no solutions where \( p > 3, p + 4^n \) are primes and \( n \) is natural number \( y \) and \( z \) are non-negative integers.

Proof: Since \( 1 + (p + 4^n)^y = z^2 \), \( z \) is even and so \( z^2 \equiv 0 \) (mod 4). Since \( p > 3 \) and \( p + 4^n \) are primes, \( p \equiv 1 \) (mod 4) or \( p \equiv -1 \equiv 3 \) (mod 4).

Case I: For \( p \equiv 1 \) (mod 4), \( 1 + (p + 4^n)^y \equiv 2 \) (mod 4) which is a contradiction since \( z^2 \equiv 0 \) (mod 4).

Case II: Suppose \( p \equiv -1 \) (mod 4).

If \( y = 2s, s \geq 1 \), then \( 1 + (p + 4^n)^y \equiv 2 \) (mod 4), which is a contradiction since \( z^2 \equiv 0 \) (mod 4).

If \( y = 2s + 1, s \geq 0 \), then \( 1 + (p + 4^n)^y = z^2 \) or equivalently
\( (p + 4^n)^{2s+1} = (z - 1)(z + 1) \). Thus there exist non-negative integers \( \alpha, \beta \) such that
\( (p + 4^n)\alpha = z + 1 \) and \( (p + 4^n)\beta = z - 1 \), where \( \alpha > \beta \) and \( \alpha + \beta = 2s + 1 \). Therefore
\( (p + 4^n)^{\beta}(p + 4^n)^{\alpha - \beta - 1} = 2 \) This implies that \( \beta = 0 \) and \( (p + 4^n)^{2s + 1} - 1 = 2 \).
Then \( (p + 4^n)^{2s + 1} = 3 \), which is impossible.

Hence the Diophantine equation \( 1 + (p + 4^n)^y = z^2 \) has no solutions where \( p > 3, p + 4^n \) are primes and \( n \) is natural number \( y \) and \( z \) are non-negative integers.

Lemma 2.2. The Diophantine equation \( p^x + 1 = z^2 \) has no solutions where \( p > 3, p + 4^n \) are primes and \( n \) is natural number \( x \) and \( z \) are non-negative integers.

Proof: Since \( p^x + 1 = z^2 \), \( z \) is even and so \( z^2 \equiv 0 \) (mod 4).
Since \( p > 3, p \) is prime, \( p \equiv 1 \) (mod 4) or \( p \equiv -1 \equiv 3 \) (mod 4).

Case I: Suppose \( p \equiv 1 \) (mod 4).

then \( p^x + 1 \equiv 2 \) (mod 4) which is a contradiction since \( z^2 \equiv 0 \) (mod 4).

Case II: Suppose \( p \equiv -1 \) (mod 4).

If \( x = 2k, k \geq 1 \), then \( p^x + 1 \equiv 2 \) (mod 4), which is a contradiction since \( z^2 \equiv 0 \) (mod 4).

If \( x = 2k + 1, k \geq 0 \), then \( p^{2k+1} + 1 = z^2 = (z + 1)(z - 1) \). Thus there exist non-negative integers \( \alpha, \beta \) such that \( p^n = z + 1 \) and \( p^\beta = z - 1 \), where \( \alpha > \beta \) and \( \alpha + \beta = 2k + 1 \). Therefore, \( p^\beta(p^{n - \beta} - 1) = 2 \). This implies that \( \beta = 0 \) and \( p^{2k+1} - 1 = 2 \). Then \( p^{2k+1} = 3 \) which is impossible.
On the Non-Linear Diophantine Equation \( p^x + (p + 4^n)^y = z^2 \) where \( p \text{ and } p + 4^n \text{ are primes} 

Hence the Diophantine equation \( p^x + 1 = z^2 \) has no solutions where \( p > 3 \), \( p + 4^n \text{ are primes and } n \text{ is natural number } y \text{ and } z \text{ are non-negative integers.} 

3. Main theorem

Theorem 3.1. The Diophantine equation \( p^x + (p + 4^n)^y = z^2 \), where \( p > 3 \), \( p + 4^n \text{ are primes and } n \text{ is natural number } y \text{ and } z \text{ are non-negative integers.} 

Proof: Since \( p^x + (p + 4^n)^y = z^2 \), \( z \) is even and so \( z^2 \equiv 0 (mod \ 4) \).

Since \( p > 3 \), \( p + 4^n \text{ are primes, } p \equiv 1 (mod \ 4) \) or \( p \equiv -1 \equiv 3 (mod \ 4) \).

Then we consider in 4 cases as follows;

Case 1. If \( x = 0 \), \( y = 0 \), then \( z^2 = 2 \) which is impossible.

Case 2. If \( x = 0 \), \( y \geq 1 \), then \( 1 + (p + 4^n)^y = z^2 \) which has no solution by Lemma 2.1.

Case 3. If \( y = 0 \), \( x \geq 1 \), then \( p^x + 1 = z^2 \) which has no solution by Lemma 2.2.

Case 4. If \( x \geq 1 \), \( y \geq 1 \), then \( p^x + (p + 4^n)^y = z^2 \) has no solutions.

We consider in 4 subcases as follow;

Subcase 1: If \( x = 2k \), \( k \geq 1 \) and \( y = 2s, s \geq 1 \), then

\( p^x + (p + 4^n)^y \equiv 2 (mod \ 4) \). which is a contradiction since \( z^2 \equiv 0 (mod \ 4) \).

Subcase 2: If \( x = 2k + 1 \), \( k \geq 0 \), \( y = 2s + 1, s \geq 0 \), then

\( p^x + (p + 4^n)^y \equiv 2 (mod \ 4) \). which is a contradiction since \( z^2 \equiv 0 (mod \ 4) \).

Subcase 3: If \( x = 2k + 1 \), \( k \geq 0 \), \( y = 2s, s \geq 1 \), then \( p^x + (p + 4^n)^y = z^2 \). Thus there exist non-negative integers \( \alpha, \beta \) such that

\( p^\alpha = z + (p + 4^n)^s \) and \( p^\beta = z - (p + 4^n)^s \), where \( \alpha > \beta \) and \( \alpha + \beta = 2k + 1 \). Therefore

\( p^\beta (p^{\alpha - \beta} - 1) = 2(p + 4^n)^s \) This implies that \( \beta = 0 \) and \( p^{2k+1} - 1 = 2(p + 4^n)^s \).

For \( k = 0 \), we obtain \( p - 1 = 2(p + 4^n)^s \). Or \( p = 2(p + 4^n)^s + 1 \), which is impossible.

For \( k \geq 1 \), we have \( p^{2k+1} - 1 = 2(p + 4^n)^s = (p - 1)(p^{2k} + p^{2k-1} + \ldots + 1) \).

It follows that \( p - 1 \) is an even positive divisor of \( 2(p + 4^n)^s \), that is \( p - 1 = 2(p + 4^n)^j \)

where \( j \) is an integer such that \( 0 \leq j < s \). For \( j = 0 \), \( p = 3 \) which contradicts the fact that \( p > 3 \).

For \( 1 \leq j < s \), we obtain \( 2(p + 4^n)^j = (p + 4)^5 \) or \( 2(p + 4^n)^j + 5 = p + 4 \) which is impossible.

Therefore, the Diophantine equation \( p^x + (p + 4^n)^y = z^2 \), where \( p > 3 \) and \( p + 4^n \text{ are primes, has no solutions.} 

Subcase 4: If \( x = 2k \), \( k \geq 1 \), \( y = 2s + 1, s \geq 0 \), then \( p^x + (p + 4^n)^y = z^2 \). Thus there exist non-negative integers \( \alpha, \beta \) such that \( (p + 4^n)^\alpha = z + p^k \) and

\( (p + 4^n)^\beta = z - p^k \), where \( \alpha > \beta \) and \( \alpha + \beta = 2s + 1 \). Therefore

\( (p + 4^n)^\beta ((p + 4^n)^{\alpha - \beta} - 1) = 2p^k \) This implies that \( \beta = 0 \) and \( (p + 4^n)^\alpha - 1 = 2p^k, (p + 4^n)^{2s+1} - 1 = 2p^k \).

For \( s = 0 \), \( (p + 4^n) - 1 = 2p^k \). Or \( (2p^{k-1} - 1) = 4^n - 1 = 3(4^{n-1} + 4^{n-2} + \ldots + 1) \). which is impossible.

For \( s \geq 1 \), we have \( (p + 4^n)^{2s+1} - 1 = 2p^k \). \( 2p^k = (p + 4^n - 1)((p + 4^n)^{2s} + (p + 4^n)^{2s-1} + \ldots + (p + 4^n) + 1) \).

It follows that \( p + 4^n - 1 \) is an even positive divisor of \( 2p^k \), that is
The non-solvability of the Diophantine equation \( p + 4^n - 1 = 2(p + 4^n)^j \), where \( j \) is an integer such that \( 0 \leq j < s \). For \( j = 0 \), \( p + 4^n = 3 \) which is impossible. For \( 1 \leq j < s \), we obtain \( p + 4^n - 1 = 2(p + 4^n)^j \). Or \( p + 4^n = 2(p + 4^n)^j + 1 \) which is impossible. Therefore, the Diophantine equation \( p^x + (p + 4^n)^y = z^2 \) has no solutions, where \( p > 3 \) and \( p + 4^n \) are primes, \( n \) is a natural number, \( x \), \( y \) and \( z \) are non-negative integers.

**Corollary 3.1.1.** The Diophantine equation \( p^x + (p + 4^n)^y = u^{2n} \) has no solution., where \( u, x, y \) and \( z \) are non-negative integers and \( n \) is a natural number.

**Proof:** Let \( u^n = z \) then \( p^x + (p + 4^n)^y = u^{2n} = z^2 \), which has no solution by Theorem 3.1.

**Corollary 3.1.2.** The Diophantine equation \( p^x + (p + 4^n)^y = u^{2n+2} \) has no solution, where \( u, x, y \) and \( z \) are non-negative integers and \( n \) is a natural number.

**Proof:** Let \( u^{n+1} = z \) then \( p^x + (p + 4^n)^y = u^{2n+2} = z^2 \), which has no solution by Theorem 3.1.

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