Annals of Pure and Applied Mathematics
Vol. 24, No. 1, 2021, 1-10
ISSN: 2279-087X (P), 2279-0888(online)
Published on 10 July 2021
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v24n1a01828

Annals of
Pure and Applied Mathematics

Diametral Covering Number of a Graph

Medha Itagi Huilgol[1*] **and** ***Kiran S***[2]

[1]Department of Mathematics, Bengaluru City University
Central College Campus, Bengaluru-560001
Email: medha@bub.ernet.in
[2]Department of Mathematics, Bangalore University
Central College Campus, Bengaluru-560001
Email: kiran.sabbanahalli@gmail.com

Received 15 May 2021; accepted 29 June 2021

Abstract. In this paper, we introduce the diametral covering number of a graph. A subset S of $V(G)$ is said to be a diametral cover for G if every diametral path of G contains at least one vertex of S. The minimum cardinality of S taken over all diametral covers is called the diametral covering number of G and is denoted by $\sigma_d(G)$. Here we have given the diametral covering number of several classes of graphs and have given bounds for the same in terms of basic graph parameters. Also, a characterization of graphs having particular diametral covering number is given.

Keywords: Diametral paths, vertex covers, Diametral covering number

AMS Mathematics Subject Classification (2010): 14H30

1. Introduction

Covering problem in graph theory is not new. There are different types of covers that exist in literature, vertices to cover vertices or edges, or vice-versa. A lucid survey due to Manlove [8] lists all of them. Precisely, the covers give rise to many famous graph parameters, viz., vertex domination, edge domination, vertex cover, edge cover, etc., [1, 3, 4, 9, 11]. The theory of domination needs no introduction. Volumes of monographs and scores of papers are available in literature. The most famous monographs are due to Haynes et al. [6, 7]. We know that a vertex cover of a graph is a set of vertices that includes at least one endpoint of every edge of the graph. It is a classical optimization problem to find a minimum vertex cover and is a typical example of an *NP*-hard optimization problem. One of Karp's 21 *NP*-complete problems is the decision version of the minimum vertex cover, and is therefore a classical *NP*-complete problem in computational complexity theory. The minimum vertex cover problem can be formulated as a half-integral linear program whose dual linear program is the maximum matching problem.

On the other hand, an edge cover is a set of edges that covers all the vertices in a graph. These were first considered by Norman and Rabin [10]. The algorithmic complexity issues of both vertex and edge covers are considered by Fernau and Manlove [4], where the clustering properties are addressed algorithmically.

The subclass of covering problems consisting of path coverings are the geodesic covering problem, the induced path covering problem, etc. All these coverings that deal with paths, in particular those involving geodesics were found to have applications in optimal transport flow in social networks. A related problem called strong edge geodetic problem was introduced and studied by Manual et al. [9].

Inspired by all these here we define and study a parameter called "diametral covering number" of a graph. This parameter originates from practical situation, like most of the covering problems. This caters to the needs to cover the longest paths, the diametral paths in a graph, so as to cover the far off nodes and the roads connecting them in any network. The study of vertices in any diametral cover may be viewed as utility centers, such as, fuel station, service station, gas station, etc. In any transport system prevailing in a particular region, where vehicles travel from one end to the other end, these utility centers need to be accessed by all these diametral paths. Hence, any transport operator always looks to minimize cost and maximize utility, in establishing the utility centers. Addressing such situations, we define the diametral cover of a graph wherein the peripheral vertices may be considered as far end origins and destination and vertices in the diametral cover, where one can establish utility centers. We formally define it in this paper and consider bounds for the same. Some characterizations are also found. But first some preliminaries are considered.

2. Basics

All the graphs considered in this paper are simple, finite, and undirected. For all the undefined terms, the reader is referred to Buckley and Harary [2] and Harary [5].

Definition 2.1. [5] *A vertex is said to cover an edge if it is incident with that edge. A set of vertices which covers all the edges of a graph G is called a vertex cover for G. The smallest number of vertices in any vertex cover for G is called its vertex covering number and is denoted by* $\alpha_0(G)$.

Definition 2.2. [5] *An edge is said to cover a vertex if it is an end vertex of that edge. A set of edges which covers all the vertices of a graph G is called an edge cover for G. The smallest number of edges in any edge cover for G is called its edge covering number and is denoted by* $\alpha_1(G)$.

Definition 2.3. [5] *A set of vertices in G is said to be independent if no two of them are adjacent. The largest number of vertices in such a set is called the vertex independence number of G and is denoted by* $\beta_0(G)$.

Definition 2.4. [5] *A set of edges in G is independent if no two of them are incident. The* *largest number of edges in such a set is called the edge independence number of G and is denoted by $\beta_1(G)$.*

Definition 2.5. [2] *The distance between two vertices in a graph is the number of edges in a shortest path connecting them. This is also called as the geodesic distance. Then length of the maximum geodesic in a given graph is called the diameter, denoted by d. The length of the minimum geodesic in a given graph is called the radius, denoted by r.*

Definition 2.6. [2] *The eccentricity $e(v)$ of a vertex v is the greatest distance between v and any other vertex, that is $e(v) = \min_{u\in v} d(u, v)$.*
Definition 2.7. [2] *A peripheral vertex in a graph of diameter d is one that is distance d from some other vertex, that is, a vertex that achieves the diameter. Formally, v is peripheral if $e(v) = d$.*

Definition 2.8. [2] *A geodetic cover of G is a set $S \subset V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The geodetic number $gn(G)$ of G is the minimum order of its geodetic covers, and any cover of order $gn(G)$ is a geodetic basis.*

3. Diametral covering number of a graph

In this section we formally introduce the concept of diametral cover.

Definition 3.1. *A subset S of $V(G)$ is said to be a diametral cover in a graph G if every diametral path of G contains at least one vertex of S. The minimum cardinality of S taken over all diametral covers is called the diametral covering number of G and is denoted by $\sigma_d(G)$.*

Remark 1. It is clear from the definition that *S* cannot be empty, that is $\sigma_d(G) \neq 0$, as the vertices of the periphery of *G* can cover at least one diametral path at a time. In cases where non-peripheral vertices of *G* do not explicitly exist to cover a diametral path, other than the peripheral vertices (of course), then a diametral cover contains the vertices of periphery of *G*, $P(G)$. itself. Here also we consider the minimum cardinality of such a cover *S*.
Alternately we can define the diametral cover as follows:

Definition 3.2. Let $I_G^d(u, v) = \{w \in V(G) / w$ lies on a shortest u, v path in G, for any two peripheral vertices $u, v\}$. If $w \in I_G^d(u, v)$, then $diam(G) = d(u, w) + d(w, v)$. Hence $S = \{w / w \in I_G^d(u, v), u, v \in P(G)\}$ is a diametral cover of G.

The minimum cardinality of *S* taken over all diametral covers is called the diametral covering number of *G* and is denoted by $\sigma_d(G)$.

Remark 2. If no such w exists, then it is clear that either $u = w$ or $w = v$.

Example 3.1.

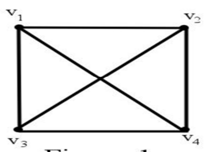

Figure 1

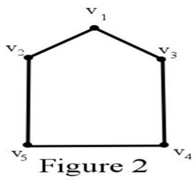

Figure 2

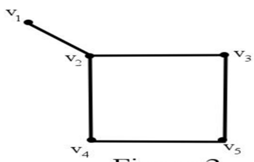

Figure 3

In Figure 1, we have $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_1, v_2, v_4\}$, $S_3 = \{v_1, v_3, v_4\}$ and $S_4 = \{v_2, v_3, v_4\}$. It has four diametral covers of minimum cardinality three. Therefore, $\sigma_d(G) = 3$.

In Figure 2, we have $S_1 = \{v_1, v_4\}$, $S_2 = \{v_1, v_5\}$, $S_3 = \{v_2, v_4\}$, $S_4 = \{v_2, v_3\}$ and $S_5 = \{v_3, v_5\}$. It has five diametral covers of minimum cardinality two. Therefore. $\sigma_d(G) = 2$.

In Figure 3, we have $S_1 = \{v_2\}$ and $S_2 = \{v_3, v_4\}$. It has two diametral covers. The minimum cardinality of diametral covers is one. Therefore, $\sigma_d(G) = 1$.

In the first section we obtain diametral covering number of certain familiar class of graphs, namely, K_n-complete graph, C_n-cycle, P_n-path, T-tree, $K_{m,n}$-complete bipartite graph, $\overline{K_{m_1}} + \overline{K_{m_2}} + \overline{K_{m_3}} + \cdots + \overline{K_{m_d}}$ -graph Also we show existence of a graph with given diametral covering number. We find upper and lower bounds for $\sigma_d(G)$, in case of unicyclic graphs in terms of the length of the cycle it contains. Lastly, we find upper and lower bounds for size of a graph, in terms of $\sigma_d(G)$, $d(G)$ and order n.

Theorem 3.1. *In any connected graph G, if $\sigma_d(G) = 1$, then G has a cut vertex.*

Proof: Let us consider a graph G with $\sigma_d(G) = 1$. Hence, there exists a vertex, say v, which is on every diametral path, joining any two peripheral vertices, say u and w. Then there cannot be a diametral path joining these vertices in $G-v$. Then $G-v$ is disconnected, so v is a cut vertex of G.

Remark 3. Converse need not be true, which follows from the example below.

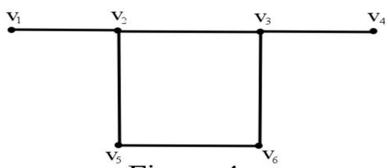

Figure 4

In Figure 4, v_2 and v_3 are cut vertices but, $S = \{v_2, v_3\}$ is a diametral cover of cardinality two. Therefore, $\sigma_d(G) = 2$.

Lemma 3.1. *For any tree T, $\sigma_d(T) = 1$.*

Proof: Consider any tree T on n-vertices. In any tree, every pair of peripheral vertices is joined by unique path (of length diameter). And each diametral path passes through center of tree. Hence one central vertex is enough to cover all diametral paths. Hence $\sigma_d(T) \leq 1$. But, $\sigma_d(T) \geq 1$, holding the equality.

Corollary 3.1. *For any path P_n, $\sigma_d(P_n) = 1$.*

Lemma 3.2. For any complete graph K_n, on n-vertices, $\sigma_d(K_n) = n - 1$.
Proof: Since $diam(K_n) = 1$, all vertices are in the periphery of G. So we need at least $n - 1$ vertices to cover all diametral paths.

Lemma 3.3. *For a cycle C_n, $\sigma_d(C_n) = 2$.*
Proof: We give proof for cycles depending upon the length of the cycle.
Case (i): Let the cycle be of even length.

Consider any vertex, say u_1, of C_n. The eccentricity of u_1, $e(u_1)$ is equal to the diameter of C_n and the vertex at distance, $diam(C_n) = \frac{n}{2}$, from u_1 to $u_{\frac{n}{2}+1}$ There are two diametral paths joining these two vertices. One is $u_1, u_2, \ldots, u_{\frac{n}{2}}, u_{\frac{n}{2}+1}$ and another is $u_1, u_n, \ldots, u_{\frac{n}{2}+2}, u_{\frac{n}{2}+1}$. So two of n-diametral paths contain u_1.
So $u_1 \in S$.
Now consider the diametral path $u_n, u_1, u_2, \ldots, u_{\frac{n}{2}}$; $u_{n-1}, u_n, u_1, u_2, \ldots, u_{\frac{n}{2}-3}, u_{\frac{n}{2}-2}, u_{\frac{n}{2}-1}$; $u_{n-2}, u_{n-1}, u_n, u_1, u_2, \ldots, u_{\frac{n}{2}-3}, u_{\frac{n}{2}-2}$;…, $u_3, u_2, u_1, \ldots, u_{\frac{n}{2}+3}$ and $u_2, u_1, u_n, \ldots, u_{\frac{n}{2}+1}, u_{\frac{n}{2}+2}$. Note that all these paths contain u_1. Note that all these paths $u_n, u_{n-1}, u_{n-2}, \ldots, u_{\frac{n}{2}+1}, u_{\frac{n}{2}}, u_{\frac{n}{2}-1}$; $u_{n-1}, u_{n-2}, \ldots, u_{\frac{n}{2}+1}, u_{\frac{n}{2}}, u_{\frac{n}{2}-1}, u_{\frac{n}{2}-2}$;…, $u_{\frac{n}{2}+3}, u_{\frac{n}{2}+2}, u_{\frac{n}{2}+1}, \ldots, u_4, u_3$ and $u_{\frac{n}{2}+2}, u_{\frac{n}{2}+1}, \ldots, u_4, u_3, u_2$; do not have u_1. Hence $\sigma_d(C_n) > 1 \Rightarrow \sigma_d(C_n) \geq 2$.

But the second set of diametral path have $u_{\frac{n}{2}+1}$ in common. So to cover all diametral paths we need to have $u_{\frac{n}{2}+1}$ in S. Hence $u_{\frac{n}{2}+1} \in S$ and $S = \{u_1, u_{\frac{n}{2}+1}\}$ giving $\sigma_d(C_n) \leq 2$. Hence equality holds.

Case (ii): Let the cycle be of odd length.

Consider any vertex, say u_1, of C_n. The eccentricity of u_1, $e(u_1)$ is equal to the diameter of C_n and there are two eccentric vertices for u_1 namely, $u_{\lfloor\frac{n}{2}\rfloor}$ and $u_{\lfloor\frac{n}{2}\rfloor+1}$. So there are two diametral paths joining these two vertices to u_1, i.e., $u_1, u_2, u_3, \ldots, u_{\lfloor\frac{n}{2}\rfloor-1}, u_{\lfloor\frac{n}{2}\rfloor}$ and $u_1, u_n, \ldots, u_{\lfloor\frac{n}{2}\rfloor+2}, u_{\lfloor\frac{n}{2}\rfloor+1}$.
Now consider the diametral paths $u_n, u_1, u_2, \ldots, u_{\lfloor\frac{n}{2}\rfloor-2}, u_{\lfloor\frac{n}{2}\rfloor-1}$; u_{n-1},

$u_n, u_1, u_2, \dots, u_{\lfloor \frac{n}{2} \rfloor -3}, u_{\lfloor \frac{n}{2} \rfloor -2}; \dots ; u_3, u_2, u_1, \dots, u_{\lfloor \frac{n}{2} \rfloor +4}, u_{\lfloor \frac{n}{2} \rfloor +3}$ and $u_2, u_1, u_n, \dots, u_{\lfloor \frac{n}{2} \rfloor +3}, u_{\lfloor \frac{n}{2} \rfloor +2}$. Note that all these paths have u_1 common.

On the other hand, the paths $u_n, u_{n-1}, u_{n-2}, \dots, u_{\lfloor \frac{n}{2} \rfloor +1}, u_{\lfloor \frac{n}{2} \rfloor}$; $u_{n-1}, u_{n-2}, \dots, u_{\lfloor \frac{n}{2} \rfloor +2}, u_{\lfloor \frac{n}{2} \rfloor +1}; \dots; u_{\lfloor \frac{n}{2} \rfloor +2}, u_{\lfloor \frac{n}{2} \rfloor +1}, u_{\lfloor \frac{n}{2} \rfloor}, \dots, u_4, u_3$ and $u_{\lfloor \frac{n}{2} \rfloor +1}, u_{\lfloor \frac{n}{2} \rfloor}, \dots, u_4, u_3, u_2$; do not have u_1.

Hence $\sigma_d(C_n) > 1 \Rightarrow \sigma_d(C_n) \geq 2$.

But this set contains both $u_{\lfloor \frac{n}{2} \rfloor +2}$ and $u_{\lfloor \frac{n}{2} \rfloor +1}$ is common. So at least one must be included in *S*, for minimality. Hence $S = \{u_1, u_{\lfloor \frac{n}{2} \rfloor +2}\}$ or $S = \{u_1, u_{\lfloor \frac{n}{2} \rfloor +1}\} \Rightarrow \sigma_d(C_n) \leq 2$.

Holding equality, and proving the result.

Proposition 3.1. *For any positive n, there exists a graph G, with* $\sigma_d(G) = n$ *and diameter d.*

Proof: Let n be any positive integer. Let G be a connected graph with u and v as peripheral vertices. Since we are constructing G with $\sigma_d(G) = n$, there are at least n distinct diametral paths between u and v. We consider $d - 1$ sets, $V_1, V_2, \cdots, V_{d-1}$ of n vertices each. Let $V_0 = \{u\}$ and $V_d = \{v\}$. In G, any two vertices, say x and y are adjacent if x and y belong to two consecutive sets V_i of $v(G)$, that is, if $x \in V(G)$, say then $xy \in E(G)$ if and only if either $y \in V_{i-1}$ or $y \in V_{i+1}$. The longest path will be between u and v of length diameter, $d(G) = d$. Let u be adjacent to all vertices of V_1 and all vertices of V_{d-1} be adjacent to v.

Hence, there are at least n distinct diametral paths between u and v, of the type $u - w^j_i - v$, where $w_i^j \in V_i$, $1 \leq i \leq d-1$, $1 \leq j \leq n$. So we need at most n vertices to cover all diametral paths, implying $\sigma_d(G) \leq n$. Any one set V_i, $1 \leq i \leq d-1$ is sufficient to cover all paths.

To show that $\sigma_d(G) \geq n$, assume, if possible, that $\sigma(G) = n - 1$. Then there exists at least one vertex, say $v_i^j \in V_i$, $1 \leq i \leq d-1$, $1 \leq j \leq n$, for which diametral paths $v_1^j v_2^j \dots v_{i-1}^j v_i^j v_{i+1}^j \dots v_{d-1}^j$ do not contain any vertex of S, contradicting the definition of the diametral cover. Hence any vertex of S, contradicting the definition of the diametral cover. Hence $\sigma_d(G) \geq n$, and proving equality.

Proposition 3.2. For any unicyclic graph G, $1 \leq \sigma_d(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$, where n is the length of the cycle of G.

Proof: Let G be a unicyclic graph of order p. Let C_n be the cycle of G. We proceed the proof by considering vertices at which paths are attached.

If only one vertex, say u_1, of C_n has a path starting from it (except cyclic adjacencies). This path is of length $p - n$ on $p - n$ vertices. Let v_1 be adjacent to u_1 and consequently to $v_2, v_3, \cdots, v_{p-n-1}, v_{p-n}$. In G, each diametral path contains this path of length $p - n$, as a sub-path and hence u_1 is sufficient to cover all diametral paths. Since $\sigma_d(G) \geq 1$ for any G, proving the first inequality.

If there exist diametral paths all of which consist sub-paths attached to vertices of the cycle, then we need at least half the vertices of the cycle to cover all diametral paths. Hence if n is even then $1 \leq \sigma_d(G) \leq \frac{n}{2}$ and if n is odd then $1 \leq \sigma_d(G) \leq \frac{n+1}{2}$, is $1 \leq \sigma_d(G) \leq \left\lceil \frac{n}{2} \right\rceil$.

Both bound are attainable, lower bound is already discussed in the theorem. The upper bound is attained by corona of a cycle C^+.

Proposition 3.3. *For a graph* $G = \overline{K_{m_1}} + \overline{K_{m_2}} + \overline{K_{m_3}} + \cdots + \overline{K_{m_d}}$, $\sigma_d(G) = \min_{2 \leq i \leq d-1}\{m_i\}$, where m_i are the order of the partite sets of G.

Proof: Clearly, by the structure of the graph, the diametral paths are between vertices of V_{m_1} and V_{m_d}. And all these diametral paths contain all vertices of $V_{m_2}, V_{m_3}, \cdots, V_{m_{d-1}}$. Hence

$$\sigma_d(G) \leq \min_{2 \leq i \leq d-1}\{m_i\} \tag{3.1}$$

Now, let us denote the vertices of $V_{m_i} = \{v_{m_i}^1, v_{m_i}^2, \dots, v_{m_i}^{m_i-1}, v_{m_i}^{m_i}\}$ where $1 \leq i \leq d$, d = *diam* (G). Let m_k be the minimum index among m_i. If possible, let us assume that a diametral cover contains less vertices than m_k. Hence, there exists at least one vertex, say $v_{m_k}^j$, of V_{m_k} which does not belong to any diametral cover. Now there exists a path starting from a vertex of V_{m_1}, say $P_1 = \{v_{m_1}^1, v_{m_2}^1, \dots, v_{m_{k-1}}^1, v_{m_k}^j, v_{m_{k+1}}^1, \dots, v_{m_{d-1}}^1, v_{m_d}^1\}$ of length $d = diam(G)$. So P_1 is a diametral path which contains $v_{m_k}^j$, a contradiction. So there cannot be any vertex left out of V_{m_k} to form S, otherwise there will be diametral paths which do not contain vertices of S. Hence

$$\sigma_d(G) \geq \min_{2 \leq i \leq d-1}\{m_i\} \tag{3.2}$$

The result follows from (3.1) and (3.2).

The following corollary gives the diametral covering number of a complete bipartite graph.

Corollary 3.2. *For a complete bipartite graph $K_{m,n}$, $\sigma_d(K_{m,n}) = min\{m, n\}$.*

Proposition 3.4. Let G be a connected graph with $\sigma_d(G) = k$. Let the number of peripheral vertices of G be m, then for the complement $\overline{G}$ of G, we have $\sigma_d(\overline{G}) \leq m$.

Proof. Given G a connected graph with $\sigma_d(G) = k$. Let us denote the set of peripheral vertices by P, hence $|P| = m$. Since $\sigma_d(G) = k$, there are at least k-disjoint diametral paths. But $P(G)$ induces a connected subgraph in $\overline{G}$. Hence at most, the number of peripheral vertices in $\overline{G}$ are $p - m$ and each of m vertices lies on at most on two of the paths joining $p - m$ vertices. Since these $p - m$ vertices are possible peripheral vertices in $\overline{G}$, the number of vertices needed to cover all diametral paths is at most m. Hence $\sigma_d(\overline{G}) \leq m$.

Theorem 3.2. For any graph G of order p, diameter d, diametral covering number n, size q, the following holds for the size of G:

$$nd \leq q \leq d - 4 + \frac{n^2 + 3n}{2} + \frac{(p-n-d+1)(p-n-d+4)}{2}.$$

Both bounds are attainable.

Proof: Let G be a graph of order p, size q, $diam(d) = d$, $\sigma_d(G) = n$. Since $\sigma_d(G) = n$, there are at least n distinct (except at diametral vertices) diametral paths. Hence we must have at least nd edges, for a graph with $\sigma_d(G) = n$, $diam(G) = d$. Hence $q \geq nd$, establishing the lower bound.

For the upper bound, keeping $\sigma_d(G) = n$, $diam(G) = d$, in mind one can distribute $p - n$ vertices in d-(sub)sets, apart from S, a minimum diametral cover. Thus we can have a partition of $V(G)$ into S and other d- subsets. Let us denote these sets as A_i, $1 \leq i \leq d$, and we note $|A_i| = k_i$, $1 \leq i \leq d$. This partition is possible s each A_i contains at least one vertex to maintain the $diam(G) = d$, that is, $k_i \geq 1$. Since S is a minimum diametral cover, $k_i \geq n$, $1 \leq i \leq d$. So G will have at most q edges,

$$q = \binom{n}{2} + \sum_{i=1}^{d}\binom{k_i}{2} + \sum_{i=1}^{d-1} k_i k_{i+1} + nk_i + nk_{i+1},$$

as each $\langle A_i \rangle$ would be complete, S would be complete and each pair of subsequent sets A_i, A_{i+1} would be of the sequential join type. Thus

$$q \leq \binom{n}{2} + \binom{\sum_{i=1}^{d} k_i}{2} + \sum_{i=1}^{d-1} k_i k_{i+1} + n(k_i + k_{i+1})$$

as

$$\sum_{i=1}^{d}\binom{k_i}{2} \leq \binom{\sum_{i=1}^{d} k_i}{2}.$$

Now the sum of products $\sum_{i=1}^{d-1} k_i k_{i+1}$ is maximum, if both terms of product are the same, that is, $k_i = k_{i+1}$. Keeping both in mind, we get $\sum_{i=1}^{d} k_i = p - n - (n-1)$, as we have to have at least one vertex in
each of A_i's, to maintain the diameter and $k_i = k_{i+1} = 1$ for $(d-1)$- A_i's.

$$G = \underbrace{K_1 + K_1 + \ldots + K_1}_{a-times} + K_n + \underbrace{K_1 + K_1 + \ldots + K_1}_{b-times} + K_{p-n-d-1} +$$
$$\underbrace{K_1 + K_1 + \ldots + K_1}_{c-times}$$

In all there are $d + 1$ subsets to maintain the diameter, hence there are $(d - 1)$ copies of K_1. Hence $a + b + c + 1 = d$.

So we get,

$$q \leq a - 1 + n + \binom{n}{2} + n + b - 1 + p - n - d + 1 + \binom{p-n-d+1}{2} + p - n - d + 1 + c - 1$$
$$= a + b + c - 3 + 2n + \binom{n}{2} + 2(p - n - d + 1) + \binom{p-n-d+1}{2}$$
$$= (d - 4) + 2n + 2(p - n - d + 1) + \frac{n(n-1)}{2} + \frac{(p-n-d+1)(p-n-d)}{2}$$
$$= d - 4 + \frac{n^2+4n-n}{2} + \frac{4(p-n-d+1)+(p-n-d+1)(p-n-d)}{2}$$
$$= d - 4 + \frac{n^2+3n}{2} + \frac{(p-n-d+1)(p-n-d+4)}{2}.$$

This bound is attained by the graph given above and is a maximal graph with respect to the $diam(G)$, as the only possible edges are between A_i and A_j where i and j are consecutive indices. G is the maximum graph with $d(G) = d$, $\sigma(G) = n$, having q edges,

$$q \leq d - 4 + \frac{n^2 + 3n}{2} + \frac{(p - n - d + 1)(p - n - d + 4)}{2}$$

as we have considered all conditions for each term of the summation for edges to be maximum.

The lower bound is attained by even cycles.

Next, we take note of some edge operations on the diametral covering number. The most fundamental operations being the edge deletion and contraction. We denote an edge contraction by G/e, for any edge e in G. And the next result shows the effect of edge contractions.

Theorem 3.3. *For a connected acyclic graph G,* $\sigma_d(G/e) = 1$.

Proof: Let G be a connected acyclic graph. Hence, by Lemma 2.1, it follows $\sigma_d(G) = 1$. On contraction of any edge in G, it gives the another tree, say $G' = G/e$. Therefore, $\sigma_d(G/e) = 1$.

Remark 4. (i) On contraction of any edge in T, P_n, C_n, $K_{1,n}$ and $S_{m,n}$ the diametral covering number is unaltered.

(ii) On contraction of any edge in K_n is diametral covering number is altered.

Remark 5. On deletion of any edge in C_n the diametral covering number is unaltered, whereas deletion of any edge in K_n the diametral covering number is altered.

4. Conclusion

In this paper we have introduced the diametral covering number of a graph. Here we have given exact value of diametral covering number of several classes of graphs and have given bounds for the same in terms of basic graph parameters. Also given characterization of graphs having particular diametral covering number. Several other cases are being considered in case of products of graphs, embedding questions with a particular diametral covering number, relations with other covering parameters like different types of domination number, vertex (edge) covering number, geodetic number, etc.

Acknowledgement. We express gratitude and thank the unknown referees for suggestions that helped in improving the overall presentation of the paper.

REFERENCES

1. S.N.Bansode and Y.M.Umathar**,** The minimum total edge dominating energy of a graph, *Annals of Pure and Applied Mathematics*, 17(2) (2018) 157-202.
2. F.Buckley and F.Harary, *Distance in Graphs*, *Addison Wesley Publishing House*, 1990.
3. V.Chandra and R.Murali, Edge domination in the shadow distance graph of some star related graphs, *Annals of Pure and Applied Mathematics*, 13(1) (2017) 33-40.
4. H.Fernau and D.F.Manlove, Vertex and edge covers with clustering properties: complexity and algorithms, *Journal of Discrete Algorithms*, 7 (2009) 149-167.
5. F.Harary, *Graph Theory*, *Narosa Publishing House Reprint 1988*, *Tenth Reprint*, 2001.
6. T.W.Haynes, S.T.Hedetniemi and P.J.Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York; 1998.
7. T.W.Haynes, S.T.Hedetniemi and P.J.Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, Inc., New York, 1998.
8. D.F.Manlove, On the algorithmic complexity of twelve covering and independence parameters of graphs, *Discrete Applied Mathematics*, 91 (1999) 155-175.
9. P.Manuel, Sandi Klavzar, A.Xavier, A.Arokiaraj and E.Thomas, Strong edge geodetic problem in networks, *Open Mathematics*, 15(1) (2017) 1225-1235. https://doi.org/10.1515/math-2017-0101
10. R.Z.Norman and M.O.Rabin, An algorithm for a minimum cover of a graph, *Proceedings of the American Mathematical Society*, 10 (1959) 315-319.
11. D.K.Thakkar and J.Neha, About ve-domination in graphs, *Annals of Pure and Applied Mathematics*, 14(2) (2017) 245-250.