Annals of Pure and Applied Mathematics
Vol. 24, No. 1, 2021, 27-35
ISSN: 2279-087X (P), 2279-0888(online)
Published on 7 August 2021
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v24n1a04836

Annals of
Pure and Applied
Mathematics

Analytical Solution of Time-Fractional Klien-Gordon Equation by using Laplace-Adomian Decomposition Method

***R.K.Bairwa*[1*] and *Karan Singh*[2]**

[1]Department of Mathematics, University of Rajasthan
Jaipur - 302004, Rajasthan, India. E-mail: dr.rajendra.maths@gmail.com
[2]Department of Mathematics, University of Rajasthan,
Jaipur - 302004, Rajasthan, India. E-mail: karansinghmath@gmail.com

Received 9 July 2021; accepted 5 August 2021

Abstract. In the present article, we use the Laplace-Adomian decomposition method to investigate the approximate analytical solution of linear and non-linear time-fractional Klien-Gordon equations with appropriate initial conditions. The derivatives considered herein, are taken in Caputo's sense. Analytical results obtained by the proposed method are in series form and numerical computation indicates that the procedure of the suggested technique is very simple and straightforward.

Keywords: Laplace transform; Adomian decomposition method; Klien-Gordon equations of fractional order; Caputo fractional derivative.

AMS Mathematics Subject Classification (2010): 26A33, 34A08, 44A10.

1. Introduction

In recent years, fractional calculus has grown in popularity due to its versatile use in a variety of scientific disciplines such as control engineering, electromagnetism, viscoelasticity, biology and signal processing, system identification, mathematical biology, statistics ,control theory, finance, chaos theory and fractional dynamics, and others [7, 8, 14, 20**,** 22], resulting in a large number of research papers devoted to the study of solutions of partial differential equations of fractional order.

Several analytical techniques for finding approximate analytical solutions for fractional partial differential equations and systems are introduced, including the Adomian Decomposition method (ADM) [9,18,24], the variational iteration method (VIM) [17], the homotopy analysis method (HAM) [15], homotopy perturbation method (HPM) [1], homotopy perturbation transform method (HPTM) [12,21], q-homotopy analysis transform method (q-HATM) [2], the iterative Laplace transform method (ILTM) [3,25], and others.

In 2001, Khuri [13] introduced a novel approach, the Laplace Adomian Decomposition method (LADM) to seek an approximate solution to a class of nonlinear differential equations. By now, the LADM technique has been used to solve Volterra integro-differential equations [27], Burger differential equations [19], Kundu Eckhaus differential equations [6] etc.

Jafari *et al.* firstly used the LADM approach to find an approximate analytical solution of linear and nonlinear fractional diffusion-wave equations [10]. Most recently, the LADM has been used to solve fractional Telegraph Equations [11] and fractional Zakharov-Kuznetsov equations [23]. The LADM approach is rid of any small or large parameters and has advantages over other approximation approaches such as perturbation. Unlike other analytical methods, LADM does not need discretization or linearization. Therefore, the results achieved by LADM are more efficient and realistic.

In this study, we examine the linear time-fractional Klien-Gordon equation of the following form

$$D_t^{\alpha}u(x,t)-u_{xx}(x,t)+bu(x,t)=f(x,t),\ 1<\alpha\le 2\ , \tag{1}$$

$$u(x,0)=g_1(x)\ ,\ \ u_t(x,0)=g_2(x)\ . \tag{2}$$

and the non-linear time-fractional Klien-Gordon equation of the form

$$D_t^{\alpha}u(x,t)-u_{xx}(x,t)+bu(x,t)+cg(u(x,t))=f(x,t),\ 1<\alpha\le 2, \tag{3}$$

$$u(x,0)=g_1(x)\ ,\ \ u_t(x,0)=g_2(x)\ , \tag{4}$$

where b and c are real, $g(u)$ is a non-linear function and f is a known analytic function. The fractional derivatives are considered in the Caputo sense.

The main advantage of this study is to extend the work of the Laplace-Adomian decomposition method (LADM) to derive the approximate analytical solution of linear and non-linear time-fractional Klien-Gordon equations.

2. Preliminaries

Some fractional calculus definitions and notation needed in the course of this work are discussed in this section.

(a) The fractional derivative of $u(x,t)$ in the Caputo sense is defined as [16, 20]

$$D_t^{\alpha}u(x,t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\eta)^{m-\alpha-1}u^{(m)}(x,\eta)d\eta, \qquad m-1<\alpha\le m, m\in N, \tag{5}$$

(b) The Laplace transform of a function $f(x)$, $x>0$ is defined as [26]

$$L[f(x)]=F(s)=\int_0^{\infty} e^{-sx}f(x)dx, \tag{6}$$

where s is real or complex number.

(c) The Laplace transform of the Caputo fractional derivative is defined as [16, 20]

$$L\left[D_t^{\alpha}u(x,t)\right]=s^{\alpha}L\left[u(x,t)\right]-\sum_{k=o}^{m-1}u^{(k)}(x,0)s^{\alpha-k-1}, \qquad m-1<\alpha\le m, m\in N, \tag{7}$$

where $u^{(k)}(x,0)$ is the k-order derivative of $u(x,t)$ with respect to t at $t=0$.

3. Basic idea of Laplace-Adomian decomposition method

To explain the basic idea of Laplace-Adomian Decomposition method [10], we take a general fractional partial differential equation may be written in an operator form as

$$D_t^{\alpha}u(x,t)+Ru(x,t)+Nu(x,t)=g(x,t),\quad m-1<\alpha\le m,\quad m\in N, \tag{8}$$

$$u^{(k)}(x,0)=h_k(x),\quad k=0,1,2,...,m-1, \tag{9}$$

where $D_t^{\alpha}u(x,t)$ is the Caputo fractional derivative of order α, $m-1<\alpha\le m$, defined by equation (5), R is a linear operator which might include other fractional derivatives of order less than α, N is a non-linear operator which also might include other fractional derivatives of order less than α and $g(x,t)$ is a known analytic function.

Applying the Laplace transform to equation (8), we have

$$L\left[D_t^{\alpha}u(x,t)\right]+L\left[Ru(x,t)+Nu(x,t)\right]=L\left[g(x,t)\right]. \tag{10}$$

Using the equation (7), we get

$$L[u(x,t)]=\frac{1}{s^{\alpha}}\sum_{k=0}^{m-1}s^{\alpha-1-k}u^{(k)}(x,0)+\frac{1}{s^{\alpha}}L[g(x,t)] \\ -\frac{1}{s^{\alpha}}L[Ru(x,t)+Nu(x,t)]. \tag{11}$$

Applying inverse Laplace transform to the equation (11), we obtain

$$u(x,t)=L^{-1}\left[\frac{1}{s^{\alpha}}\left(\sum_{k=0}^{m-1}s^{\alpha-1-k}u^{(k)}(x,0)+L[g(x,t)]\right)\right] \\ -L^{-1}\left[\frac{1}{s^{\alpha}}L[Ru(x,t)+Nu(x,t)]\right]. \tag{12}$$

The ADM solution $u(x,t)$ is represented by the following infinite series

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t), \tag{13}$$

and the non-linear term is decomposed as follows

$$Nu(x,t)=\sum_{n=0}^{\infty}A_n\ , \tag{14}$$

where A_n are the Adomian polynomials given by

$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0},\quad n=0,1,2,.... \tag{15}$$

Substituting equations (13) and (14) into equation (12), we get

$$\sum_{n=0}^{\infty} u_n(x,t) = L^{-1}\left[\frac{1}{s^\alpha}\sum_{k=0}^{m-1} s^{\alpha-1-k}u^{(k)}(x,0) + \frac{1}{s^\alpha}L[g(x,t)]\right] - L^{-1}\left[\frac{1}{s^\alpha}L\left(R\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + \sum_{n=0}^{\infty} A_n\right)\right]. \tag{16}$$

Using the Adomian technique, we determine the formal recurrence relations in the elegant form as

$$\left.\begin{aligned} u_0(x,t) &= L^{-1}\left[\frac{1}{s^\alpha}\sum_{k=0}^{m-1} s^{\alpha-1-k}u^{(k)}(x,0) + \frac{1}{s^\alpha}L[g(x,t)]\right], \\ u_{n+1}(x,t) &= -L^{-1}\left[\frac{1}{s^\alpha}L\left(R\left(u_n(x,t)\right) + A_n\right)\right],\ n = 0,1,2,..., \end{aligned}\right\} \tag{17}$$

In general, the solutions in the above series converge rapidly. The classical approach to convergence of this type of series has been presented by Cherruault and Adomian [4] and Cherruault *et al.* [5].

4. Implementation of the method

In this part, we use the above-mentioned reliable method to solve linear and non-linear time-fractional Klien-Gordon equations with initial conditions.

Example 1. In this example we consider the following linear time-fractional Klien-Gordon equation [24]

$$D_t^\alpha u(x,t) - u_{xx} + u = 0\ ,\ 1 < \alpha \le 2, \tag{18}$$

with the initial conditions

$$u(x,0) = 0 \text{ and } u_t(x,0) = x\ . \tag{19}$$

Taking the Laplace transform of the equation (18), we have

$$L\left[D_t^\alpha u(x,t)\right] - L\left[\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\right] = 0. \tag{20}$$

Using the fractional derivative property of the Laplace transform, we get

$$L[u(x,t)] = \frac{1}{s^\alpha}\sum_{k=0}^{2-1} s^{\alpha-1-k}u^{(k)}(x,0) - \frac{1}{s^\alpha}L\left[\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\right]. \tag{21}$$

Applying inverse Laplace transform to the equation (21), we get

$$u(x,t) = \sum_{k=0}^{2-1}\left(\frac{\partial^k u(x,t)}{\partial t^k}\right)\frac{t^k}{k!} - L^{-1}\left[\frac{1}{s^\alpha}L\left(\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\right)\right]. \tag{22}$$

Substituting the results from equations (13) and (14) in the equation (22) and applying the equation (17), we determine the components of the LADM solution as follows

$$u_0(x,t) = \sum_{k=0}^{2-1}\left(\frac{\partial^k u(x,t)}{\partial t^k}\right)_{t=0}\frac{t^k}{k!} = xt, \tag{23}$$

$$u_1(x,t) = -L^{-1}\left[\frac{1}{s^\alpha}L\left(\frac{\partial^2 u_0(x,t)}{\partial x^2}+u_0(x,t)\right)\right] = -\frac{xt^{\alpha+1}}{\Gamma(\alpha+2)}, \quad (24)$$

$$u_2(x,t) = -L^{-1}\left[\frac{1}{s^\alpha}L\left(\frac{\partial^2 u_1(x,t)}{\partial x^2}+u_1(x,t)\right)\right] = (-1)^2\frac{xt^{2\alpha+1}}{\Gamma(2\alpha+2)}, \quad (25)$$

$$u_3(x,t) = -L^{-1}\left[\frac{1}{s^\alpha}L\left(\frac{\partial^2 u_2(x,t)}{\partial x^2}+u_2(x,t)\right)\right] = (-1)^3\frac{xt^{3\alpha+1}}{\Gamma(3\alpha+2)}, \quad (26)$$

and so on. The other components may be obtained accordingly.
Thus, the solution in the series form is given by

$$u(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+...$$
$$= x\left[t-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}+...\right]. \quad (27)$$

The same result was obtained by Sharma and Bairwa [24] using ADM.
If we put $\alpha = 2$, in Eq. (27), we have

$$u(x,t) = x\sin t, \quad (28)$$

which is the exactly the same solution obtained by earlier Mohyud-Din *et al.* [17] using VIM method.

Example 2. We consider the following linear time-fractional Klien-Gordon equation [24]

$$D_t^\alpha u - u_{xx} + u = 2\sin x, \; 1<\alpha\le 2, \quad (29)$$

with the initial conditions

$$u(x,0) = \sin x \quad \text{and} \quad u_t(x,0) = 1. \quad (30)$$

Taking the Laplace transform of the equation (29), we have

$$L\left[D_t^\alpha u(x,t)\right] - L\left[\frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) + 2\sin x\right] = 0. \quad (31)$$

Using the fractional derivative property of the Laplace transform, we get

$$L[u(x,t)] = \frac{1}{s^\alpha}\sum_{k=0}^{2-1} s^{\alpha-1-k}u^{(k)}(x,0) + \frac{1}{s^\alpha}L\left[\frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) + 2\sin x\right]. \quad (32)$$

Applying inverse Laplace transform to the equation (32), we get

$$u(x,t) = \sum_{k=0}^{2-1}\left(\frac{\partial^k u(x,t)}{\partial t^k}\right)\frac{t^k}{k!} + L^{-1}\left[\frac{1}{s^\alpha}L\left(\frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) + 2\sin x\right)\right]. \quad (33)$$

Substituting the results from equations (13) and (14) in the equation (33) and applying the equation (17), we determine the components of the LADM solution as follows

$$u_0(x,t) = \sum_{k=0}^{2-1}\left(\frac{\partial^k u(x,t)}{\partial t^k}\right)_{t=0}\frac{t^k}{k!} + L^{-1}\left[\frac{1}{s^\alpha}L\left(2\sin x\right)\right]$$

$$= \sin x + t + 2\sin x \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{34}$$

$$u_1(x,t) = L^{-1}\left[\frac{1}{s^{\alpha}} L\left(\frac{\partial^2 u_0(x,t)}{\partial x^2} - u_0(x,t)\right)\right]$$

$$= -2\sin x \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - 4\sin x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{35}$$

$$u_2(x,t) = L^{-1}\left[\frac{1}{s^{\alpha}} L\left(\frac{\partial^2 u_1(x,t)}{\partial x^2} - u_1(x,t)\right)\right]$$

$$= 4\sin x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 8\sin x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \tag{36}$$

$$u_3(x,t) = L^{-1}\left[\frac{1}{s^{\alpha}} L\left(\frac{\partial^2 u_2(x,t)}{\partial x^2} - u_2(x,t)\right)\right]$$

$$= -8\sin x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - 16\sin x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \tag{37}$$

and so on. The other components may be derived likewise.
Thus, the solution in the series form is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots$$

$$= \sin x + \left[t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \ldots\right]. \tag{38}$$

The same result was obtained by Sharma and Bairwa [24] using ADM.
If we put $\alpha = 2$, in Eq. (38), we have

$$u(x,t) = \sin x + \sin t, \tag{39}$$

which is the exactly the same solution obtained by earlier Wazwaz [28] using ADM technique.

Example 3. Finally, we consider the following non-linear time-fractional Klien-Gordon equation [24]

$$D_t^{\alpha} u - u_{xx} - (u_x)^2 - u^2 = 0, \; 1 < \alpha \le 2, \tag{40}$$

with the initial conditions

$$u(x,0) = 0 \text{ and } u_t(x,0) = e^x. \tag{41}$$

Taking the Laplace transform of the equation (40), we have

$$L\left[D_t^{\alpha} u(x,t)\right] - L\left[\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + u^2\right] = 0. \tag{42}$$

Using the fractional derivative property of the Laplace transform, we get

$$L[u(x,t)] = \frac{1}{s^\alpha}\sum_{k=0}^{2-1} s^{\alpha-1-k}u^{(k)}(x,0) + \frac{1}{s^\alpha}L\left[\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + u^2\right]. \tag{43}$$

Applying inverse Laplace transform to the equation (43), we get

$$u(x,t) = \sum_{k=0}^{2-1}\left(\frac{\partial^k u(x,t)}{\partial t^k}\right)\frac{t^k}{k!} + L^{-1}\left[\frac{1}{s^\alpha}L\left(\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + u^2\right)\right]. \tag{44}$$

Substituting the results from equations (13) and (14) in the equation (44) and applying the equation (17), we determine the components of the LADM solution as follows

$$u_0(x,t) = \sum_{k=0}^{2-1}\left(\frac{\partial^k u(x,t)}{\partial t^k}\right)_{t=0}\frac{t^k}{k!} = e^x t, \tag{45}$$

$$u_{n+1}(x,t) = L^{-1}\left[\frac{1}{s^\alpha}\left(\frac{\partial^2 u_n}{\partial x^2}\right)\right] + L^{-1}\left[\frac{1}{s^\alpha}(A_n)\right], n = 0,1,2,..., \tag{46}$$

where A_n are the Adomian polynomials for the non-linear terms $Nu = \left(\frac{\partial u}{\partial x}\right)^2 + u^2$.

Now, for $n = 0,1,2,...,$ and using equations (15) and (46), we have

$$A_0 = 0\ , \tag{47}$$

$$u_1(x,t) = \frac{e^x t^{\alpha+1}}{\Gamma(\alpha+2)}\ , \tag{48}$$

$$A_1 = 0\ , \tag{49}$$

$$u_2(x,t) = \frac{e^x t^{2\alpha+1}}{\Gamma(2\alpha+2)}\ , \tag{50}$$

$$A_2 = 0\ , \tag{51}$$

$$u_3(x,t) = \frac{e^x t^{3\alpha+1}}{\Gamma(3\alpha+2)}\ , \tag{52}$$

and so on. The other components can be obtained accordingly.
Thus, the solution in the series form is given by

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ... \\ &= e^x\left[t + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + ...\right]. \end{aligned} \tag{53}$$

The same result was obtained by Sharma and Bairwa [24] using ADM.
If we put $\alpha = 2$, in Eq. (53), we have an elegant result as

$$u(x,t) = e^x \sinh t\ , \tag{54}$$

5. Concluding remarks

The Laplace Adomian decomposition technique (LADM) has been successfully used to provide an approximate analytical solution to the time-fractional Klien-Gordon equation with initial conditions. The analytical results have been presented in the form of a power series with easily computed terms. It is worth mentioning that the method is capable of decreasing the volume of computational effort when compared to classical methods while keeping the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach.

Acknowledgments. The support provided by the University Grants Commission, New Delhi through a Junior Research Fellowship (JRF) to one of the authors, Mr. Karan Singh, is gratefully acknowledged. The authors are extremely thankful to the anonymous reviewers for their remarkable comments, suggestions and ideas that helped to improve this paper.

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