C³ Rational Quintic Spline

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Abstract. This paper discuss the construction of new C³ rational quintic spline interpolation with quintic numerator and cubic denominator the idea has been extended to shape preserving interpolation for positive data using the constructed rational quintic spline interpolation. There are two parameter $v_i$ and $w_i$ play very crucial role to change shape of the resolution interpolating curves. However, as per spline interpolating with C³ continuously is not able to preserve the positive data notably our scheme is easy to use and does not requires knots insertion and C³ continuity unable to achieve by solving for the second derivations $i = 1, 2, \ldots, n - 1$. An error analysis which the function to be interpolation is $f(t) \in C^3[t_0, t_n]$.

Key words: Rational, Interpolation, quintic, cubic, shape preserving, parameters.

AMS Mathematics Subject Classification (2010): 65D07

1. Introduction

Lower order Rational Spline have some weakness such that the interpolating curves may be give for unwanted behaviour of the original data due to wiggles along some interval. Thus uncacker behaviour may destroy the data, data are wastage. For some application any negative values is unacceptable. In my discovery any important information that may exist in the original data. Due to the fact that lower order spline is not able to produce completely the positive, monotonocity and convex interpolation curves on entire given interpolation. Fritch and Carlson [1] have discussed the monotonicity, positivity. Preserving by using cubic spline interpolation by modifying the first derivative values in which the shape valuation is found. Butt and Brodlie [19] have used cubic spline interpolation to preserve the positivity and convexity of the finite data by inserting extra knots in the interval in which the positively and/or convexity is not preserved by the cubic spline. There is no controlling to the final shape of the interpolating curves first user change data then correction will be final shape of the interpolation curve is possible the alteration of the first derivative parameters is require in the method. Sarfraz et al [6, 7] and Hussain and M. Hussain [9] studied positive preserving for curves and statistics by utilizing rational cubic spline with quadratic denominator. In works of by Hussain et al
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[9] and Sarfraz et al [11] the rational cubic spline with quadratic denominator has been used for positivity, monotonicity and convexity preserving with $C^2$ continuity, Hussain [10] have only one free parameter meanwhile Sarfaraz et al [12] have no free parameter.

Abbass [13, 14, 15] have discussed the positivity by using new $C^2$ rational cubic spline with two free parameter. Delbourgo [16], Gregory [8], and Karim and Kong [19] have proposed new $C^1$ rational cubic spline (cubic/quadratic) with three parameters in which two parameter are free parameter. Hussain et al [2009] proposed a piecewise $C^2$ rational quintic function with two free parameters for the interpolation of curve data. In numerical result effect of free parameter is shown. $C^2$ rational quintic function change to the to the non-trivial polynomial curve by taking very large and small values of free parameters. Hussain et al (2014) device a Oval quadratic trigonometric interpolant with two parameters for the shape. Preservation of curve data.

The proposed curve scheme is unique in its representation and it is equally applicable for the data with derivative or without derivatives. The proposed scheme is free from derivative modification for shape preservation. The proposed scheme is must be produce positive interpolation in everywhere. Meanwhile in the scheme of Hussain [10] and Hussain et al leave positivity in some subintervals.

Methodology
In this paper a $C^1$ rational with two free parameter is introduced i.e. quantic upon cubic and its convergence is analysed further error bounds obtained.

2. $C^3$ rational quintic spline function

The piecewise rational functions are in use for interpolation scalar data (Hussain and Sarfarose 2008, Sarfaroz et al 2013).

Although there rational interpolation scheme are local. The order of continuity of these scheme are local, the order of continuity of there scheme to knots to $C^1$ Duan et al (2011) $C^2$ continuity were used $C^1$ rational interpolation function is introduced here.

Let $x_i < x_{i+1}, i = 0, 1, \ldots, n-1$ the partition if interval $[0,1]$ with $x_0 = a$ and $x_n = b$. Take $f_i, d_i$ and $D_i$ are function values $D_i$ two functional values i.e.

\[ s(x_i) = f_i, \quad s(x_{i+1}) = f_{i+1}, \quad s'(x_i) = d_i, \quad s'(x_{i+1}) = d_{i+1}. \]

The purpose $C^3$ rational quintic function R2F, $s(x)$ is defined over each subinterval as :

\[ s(x) \equiv s_i(x) = \frac{P(\theta)}{q(\theta)} \]

(1)

where

\[ P(\theta) = A_0 p_1(\theta) + A_1 p_2(\theta) + A_2 p_3(\theta) + A_4 p_4(\theta) + A_5 p_5(\theta) \]

is a piecewise quintic polynomial.

where

\[
\begin{align*}
  p_1(\theta) &= (1-\theta)^3 (1+3\theta+6\theta^2) \\
  p_2(\theta) &= \theta^3 (1-15\theta+6\theta^2) \\
  p_3(\theta) &= (\theta - 6\theta^3 + 8\theta^4 - 3\theta^5) = \theta (1-\theta)(1-5\theta^2+3\theta^3) \\
  p_4(\theta) &= (-4\theta^2+7\theta^4-3\theta^5) = -\theta^2(1-\theta)(-4+3\theta) \\
  p_5(\theta) &= (1+3\theta+6\theta^2) \\
  q(\theta) &= \frac{1}{3!} (1+3\theta+6\theta^2) \\
\end{align*}
\]
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\[
P_4(\theta) = \frac{\theta^2 - 3\theta^3 + 3\theta^4 - \theta^5}{2} = \theta^3(1 - \theta)^2
\]

\[
P_6(\theta) = \frac{\theta^3(1 - \theta)^2}{2}
\]

Let \( q_i(\theta) = \left[ (1 - \theta)^2 + v_i \theta (1 - \theta)^2 + w_i \theta^2 (1 - \theta)^2 + \theta^3 \right] \) \hspace{1cm} (2)

where \( v_i \) and \( w_i \) are free parameters is cubic piecewise polynomial and

\[\theta = \frac{x - x_i}{h_i} \in [0, 1].\]

Consider rational quintic spline satisfy following interpolating conditions :-

\[s(x_i) = f(x_i), \quad S(x_{i+1}) = f(x_{i+1}), \quad S'(x_i) = f'(x_i).
\]

\[s''(x_i) = f''(x_i) \quad \text{Let} \quad s''(x_i) = d_i
\]

\[s''(x_i) \mid x_{i+1} = D_i, \quad s''(x_{i+1}) = D_{i+1} \quad (3)
\]

It follows from (2) that the rational quintic function (1) is a \( C^3 \) continuations function at knots. If \( v_i = 3, w_i = 3 \) the rational quintic function become quintic Hermite Interpolateant (Farin 2005). Rest of paper \( v_i \) and \( w_i \), are assumed to be positive read free parameters.

**Note:**

(1) If \( v_i = 1 \) and \( w_i = 1 \) the rational quintic function (1) becomes cubic Hermite interpolant (Farin 2005).

(2) Given 2D data \((f, d, D_i), i=0, 1, 2,...,n\) defined over the interval \([x_0, x_n]\) if we take \( v_i = 3, w_i = 1 \) in each subinterval and apply third order derivative continuity at knots.

By using interpolating condition we have

\[s(x_i) = h = A_0, \quad s(x_{i+1}) = f_{i+1}, \quad s(x_i) = A_4, \quad d_{i+1} = h_i + (3 + v_i) f_i = A_2
\]

\[d_{i+1} h_i + (-3 + v_i) f_i = A_3
\]

\[D_i h_i^2 + 2(-3 + v_i) d_{i+1} h_i + (6 - 4v_i + 2w_i) f_i = A_4
\]

\[A_2 = D_i h_i^2 + 2(-3 + v_i) d_{i+1} h_i + (6 - 2v_i - 4w_i) f_i = A_1
\]

So \( s(x) = \left[ f_i p_1(t) + f_{i+1} p_2(t) + \{d, h_i t(-3 + v_i) f_i \} \right] \hspace{1cm} (4)

\[f(t) + [d_{i+1} h_i + (-v_i + 3) f_i] p_4(t) + \left[ h_i^2 D_i + 2(h_i (3+1) + (6 - 4v_i + 2w_i)) f_i \right] P_6(t) / q_6(t)
\]

\[P_2(t) + [h_i^2 D_{i+1} + 2(d_{i+1} h_i (-w_i + 3)) + (6 + 2v_i - 4w_i) f_i] / P_6(t) + A_2
\]

\[f_i P_1(t) + (-3 + v_i) P_4(t) + (6 - 4v_i + 2w_i) P_5(t) + f_i + A_4
\]

\[P_2(t) + (3 - w_i) P_3(t) + (2v_i - 4w_i + 6) P_5(t)] + h_i P_2(t) + [d_i h_i P_2(t) + d_i h_i h_i P_4(t) + (h_i^2 D_i + 2(h_i h_i (-w_i + 3)) P_6(t) / q_6(t)
\]

When simple calculation given \( v_i \to 0, w_i \to 0 \)
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\[
s_i(x) = ((1-t)f_i + tf_{i+1}) + t(1-t) \left\{ \frac{[\Delta f_i(2t-1)] + d_i h_i (1-t)^2 - d_{i+1} h_i t^2 + h^2 t (1-t)}{2} \right\} D_i (1-t) + t D_{i+1} \}
\]

\[
\frac{(1-t)^3 + t^3}{(1-t)^3 + t^3}
\]

and where let \( v_i \to \infty, w_i \to \infty \) is

\[
s(x) = \left[ h_{i+1} + (1-t) f_i \right]
\]

is a linear equation which is straight line i.e. increase \( v_i \) and \( w_i \), \( s_i(x) \) reduce to straight line.

**Table 1:** Generates from constant function \( f(x) = 2 \), constant data is interpolated by the \( C^3 \)-continuous rational quintic spline (1) for different values of few parameter \( v_i \) and \( w_i \).

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
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<tbody>
<tr>
<td>t</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
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Design

<table>
<thead>
<tr>
<th>y</th>
<th>z</th>
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<tbody>
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</table>

\( x \)

**Figure 1:** \( C^3 \) Rational quintic interpolation constant from with any choice of \( v_i, w_i \).

Now using continuity of third derivatives, we obtain

\[
3(2-v_i)h_{i+1}, D_{i+1} + 9(h_i + h_{i+1})D_i + 3(2-v_i)h_{i+1}, D_{i+1} = F_i \quad i = 1,2,\ldots,n
\]

where

\[
F_i = \frac{h_i}{h_{i+1}} \left[ -60 \Delta f_i + (30 + 4v_i - 2w_i)d_{i+1} + 30d_i \right]
\]

\[
+ \frac{h_{i+1}}{h_i} \left[ 60 \Delta f_{i+1} - 24d_i - (36 - 4v_i + 2w_i)d_{i+1} \right]
\]

\[
+ \frac{h_i}{h_{i+1}} \left[ -1 + 6(-3 + v_i)f_{i+1} - 6(v_i - w_i)f_{i+1} \right]
\]

\[
+ \frac{h_i}{h_{i+1}} \left[ f_{i+1} \left( -20v_i + 10w_i + 30 \right) + (12 - 8v_i + 4w_i) f_i \right]
\]

This is the system of linear equation and satisfies diagonally dominant property and equation have unique solution for unknown \( D_i, i = 1,2,\ldots,n \), \( v_i \geq 0 \) and \( w_i \geq 0 \).

ArithmeticMean Approximation Scheme for derivatives:

The given 2D data \( \{(x_i, f_i), i = 0,1,\ldots,n\} \) defined over the interval \([a, b]\) where \( x_i < x_{i+1}, i = 0,1,\ldots,n - 1 \) the arithmetic choice of first derivatives at knots \( x_i \) are given by the following formula

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\[ d_o = \Delta_o + (\Delta_0 - \Delta_1) \frac{h_0}{h_0 + h_1} \]

\[ d_n = \Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2}) h_{n-1} (h_{n-1} + (h_{n-2}))^{-1} \]

\[ d_n = \frac{\Delta_n + \Delta_{n-1}}{2}, \quad i = 1, 2, \ldots, n - 1 \]

Hence, \( \Delta_i = \frac{f_{i+1} - f_i}{h_i} \), \( h_i = x_{i+1} - x_i \) \( i = 0, 1, \ldots, n - 1 \). (8)

Similarly the second order derivatives \( D_i \) at knots.

Geometric Mean (GMM)

Consider

\[ d_0 = \begin{cases} 0 & \text{if} \Delta_o \text{ or } \Delta_{2:0} = 0 \\ \Delta_0 \left( 1 + \frac{h_2}{h_1} \right) & \text{otherwise} \\ \Delta_0 \left( \frac{h_2}{h_1} \right) \end{cases} \]

\[ d_n = \begin{cases} 0 & \Delta_{n-1} = 0 \text{ or } \Delta_{n,n-2} = 0 \\ \delta^{n-1}_{n,n-2} & \text{otherwise} \\ \end{cases} \]

Delbergo and Gregory [17] give main details about the method that can be used to estimate the first derivative and same as second derivatives by A.M.M. we can find \( d_0 \) and \( d_n \).

Similarly the second order derivative \( D_i \) at knots \( x_i \) are given by the following formula:

\[ D_0 = M_0 + (M_0 - M_1) h_0 (h_0 + h_1)^{-1} \]

\[ D_n = M_{n-1} + (M_{n-1} - M_{n-2}) h_{n-1} (h_{n-1} + h_{n-2}) \]

\[ D_i = M_i + M_{i+1}, \quad i = 1, 2, \ldots, n - 1 \]

Here \( M_i = \frac{d_{i+1} - d_i}{2} \)

\( h_i = x_{i+1} - x_i \), \( i = 0, 1, \ldots, n - 1 \) (11)

**Remark 1:** When \( v_i = w_i = 0 \), the RQP reduce to QP without any parameter.

<table>
<thead>
<tr>
<th>Table 2: A data from work by Sarfaraz et al [6]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I )</td>
</tr>
<tr>
<td>( x_i )</td>
</tr>
<tr>
<td>( H )</td>
</tr>
<tr>
<td>( h_i )</td>
</tr>
<tr>
<td>( D_i )</td>
</tr>
<tr>
<td>( d_i(C^1) )</td>
</tr>
<tr>
<td>( d_i(C^2) )</td>
</tr>
<tr>
<td>( d_i(C^3) )</td>
</tr>
</tbody>
</table>
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Default: $C^1$ quintic spline

![Figure 2: $C^2$ quintic rational spline, $C^3$ quintic rational spline $v_j = w_i = 3$](image)

3. Positive preserving using $C^3$ rational quintic spline

In this action, the positively preserving by using the proposed $C^3$ rational quintic spline defined by will be discussed in brief. We will follow same method of Karim and Kong [12, 13] and Abbass et.al. [20, 21].

Given the strictly positive set of data $(x_i, f_i)_{i=1,2,\ldots,n}$

$x_0 < x_1 < \ldots < x_n$ such that $f_i > 0, i = 0, 1, \ldots, n$

Now the rational quintic spline will preserve of the data iff $P_i(0) > 0$ and $Q_i(0) > 0$

Since for all $i, v_i > 0, e=0, 1, \ldots, n-1$. Thus $Q_i(x) > 0$ iff $P_i(0) > 0, i = 0, 1, \ldots, n-1$.

The quintic polynomial $P_i(Q), i = 0, 1, \ldots, n$ can be written as

$$P_i(\theta) = A_i \theta^5 + B_i \theta^4 + C_i \theta^3 + D_i \theta^2 + E_i \theta + F_i$$  \hspace{1cm} (12)

Let $\theta = \frac{S}{s+1}, s \geq 0$ in $x$ and solve.

We get $P_i(\theta) = AS^3 + BS^4 + CS^3 + GS^2 + HS + F$  \hspace{1cm} (13)

where

$$A = D_{i+1}h_i^2 + t(-w_i + 3)d_{i+1}h_if_{i+1} + (6 + 2v_i - 4w_i)f_{i+1}$$

$$B = D_ih_i^2 + 2(-3 + v_i)d_ih_if_i + (6 + 2w_i - 4v_i)f_i$$

$$C = d_{i+1}h_i + (-w_i + 3)$$

$$D = d_ih_i + (-3 + v_i)$$

$$E = f_{i+1}$$

$$F = f_i$$

If $(-w_i + 3) > 0 \Rightarrow 3 \geq w_i$
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\[ v_i - 3 > 0 \Rightarrow v \geq 3 \]

**Theorem 4.** For strictly data defined in (18) C³ rational quintic spline interpolation defined over the interval \([x_0, x_n]\) is positive if in each subinterval \([x_i, x_{i+1}]\), \(i = 0, 1, \ldots, n-1\). The involving parameter \(v_i\) and \(w_i\) satisfying the following:

\[ v_i, w_i \geq 0 \]

\[ A + B + C + D + E + F > 0 \text{ iff } 3 \geq w, \ v_i \geq 3 \]

(14)

**Remark.** The sufficient condition for positive preserving by using C² rational quintic interpolant is different from sufficient condition for positive preserving by using C¹ rational quintic interpolant. To achieve C³ condition in the derivative parameter \(D_0, i=1, 2, \ldots, n-1\) must be calculated from C² condition given in (7) mean while to achieve C² condition. The second derivative \(d_i, i=0, 1, \ldots, 4\) are estimated by using standard approximation methods such as A.M.M.

**Arithmetic mean method.** Comprising to Abbas [20]. Delbourage [19], Gregery [8], Karim [12], Hussain et.al. (2011) the C² continuous and upto first derivatives obtain but in this paper C³ contemnors shape preserving and second derivative obtained also produce completely positive preserving interpolating curves with C³ continuously but in some papers the method is computationally expensive and typical Linear equations of second derivatives investigate.

**Table 3:** A positive dot a form work by Hussain et.al. [2011]

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>0</td>
<td>1.0</td>
<td>1.70</td>
<td>1.80</td>
<td>1.90</td>
</tr>
<tr>
<td>(f_i)</td>
<td>.25</td>
<td>1.0</td>
<td>11.10</td>
<td>2.5</td>
<td></td>
</tr>
</tbody>
</table>

**Table 4:** Numerical results

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d_i(C))</td>
<td>-7.296</td>
<td>8.796</td>
<td>123.429</td>
<td>154.570</td>
</tr>
<tr>
<td>(\Delta_i)</td>
<td>0.75</td>
<td>14.429</td>
<td>139</td>
<td></td>
</tr>
<tr>
<td>(v_i)</td>
<td>.5</td>
<td>.5</td>
<td>.5</td>
<td></td>
</tr>
<tr>
<td>(w_i)</td>
<td>.5</td>
<td>.5</td>
<td>.5</td>
<td></td>
</tr>
<tr>
<td>(D_i(C))</td>
<td>13.94</td>
<td>3.14</td>
<td>0.25</td>
<td></td>
</tr>
</tbody>
</table>

**Table 5:** Lead contamination drain of Chenab river. Malik and Hussain et al. ([12,13])

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>20</td>
<td>42</td>
<td>45</td>
<td>47</td>
<td>52</td>
</tr>
<tr>
<td>y</td>
<td>.05</td>
<td>.06</td>
<td>1.5</td>
<td>1.65</td>
<td>1.4</td>
</tr>
</tbody>
</table>

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Numerical Results:

\[
\begin{array}{cccccc}
C_i & d_{jC_i} & D_{C_j} & w_i & v_i \\
-0.4215 & .2402 & .2775 & .0125 & -0.1393 \\
.0456 & .0213 & -0.0600 & .0814 & .0426 \\
.0456 & 7.06 & -5.7688 & -1.1877 & .0426 \\
.005 & .005 & .005 & .005 & .005 \\
3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

Numerical result (Change \(v_i, w_i\))

\[
\begin{array}{cccccc}
v_i & 3 & 3 & 3 & 3 & 3 \\
w_i & 3 & 3 & 3 & 3 & 3 \\
D_{C_j} & .456 & 8.8367 & 0.97177 & 1.61727 \\
\end{array}
\]

Numerical Result

\[
\begin{array}{cccccccc}
v_i & 4 & 4 & 4 & 4 & 4 \\
w_i & 1 & 1 & 1 & 1 & 1 \\
D_{C_j} & .456 & 3.6358 & 2.0527 & -7.3573 \\
v_i & 3.3 & 3.3 & 3.3 & 3.3 & 3.3 \\
w_i & 2 & 2 & 2 & 2 & 2 \\
D_{C_j} & .1424 & 9.7005 & -2.9666 & 1.758 & 0.039 \\
\end{array}
\]

Table 6: Creative in Human blood

\[
\begin{array}{cccccccc}
x & 2 & 30 & 32 & 35 & 37 & 39 \\
y & 1.51 & 0.80 & 1.05 & 0.6 & 0.51 & 0.58 \\
D_{C_j} & .1424 & 10.4522 & 04.0702 & 0.12901 & 0.369 \\
v_i & 3.3 & 3.3 & 3.3 & 3.3 & 3.3 \\
w_i & 2 & 2 & 2 & 2 & 2 \\
\end{array}
\]

Numerical results (Change \(v_i, w_i\))

\[
\begin{array}{cccccc}
D_{C_j} & .1424 & .05994 & .3130 & .7616 & .0369 \\
v_i & 3 & 3 & 3 & 3 & 3 \\
w_i & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

Table 7: Data lying above line \(y=x+3\)

\[
\begin{array}{cccccccc}
x & 0 & 4 & 0.21 & 30 & 32 \\
y & 22.8 & 8 & 33.9 & 38.9 & 43.6 \\
v_i = w_i = 3 \\
D_{C_j} & 0.054 & - & 9.9258 & 36.7216 \\
\end{array}
\]

Table 8: Data lying above line \(y = 0.5x - 1\).

\[
\begin{array}{ccc}
x & 1 & 5.5 & 6 \\
y & 1 & 2 & 6 \\
\end{array}
\]
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\[ v_i = w_i = 3 \]
\[ D_i(C^3) = 87.250 \]
\[ w_j = 2.5v_i = 3.5 \]
\[ D_i(C^3) = -61.639 \]

4. Error analysis

In this section, the error Analysis for the function to be interpolated to \( f(t) \in C^3 \left[ t_0, t_n \right] \) using over C\(^3\) rational quintic spline which will discussed in detail. Note that the construction rational quintic spline which will be discussed in details. Note that the constructed rational quintic spline which will be discussed in detail. Note that the constructed rational quintic spline with two parameters is a local interpolant and without loss of generality we take interval.

\[ I = [t_i, t_{i+1}] \]

By using Perono Kernel theorem [25] the error of interpolation in each interval \( I = [t_i, t_{i+1}] \) is dedness as

\[
R[f] = f(t) - P_i(t) = \frac{1}{2} \int_{t_i}^{t_{i+1}} R \left( \frac{t - t_j}{h_j} \right)^2 dt
\]

where \( f = \left[ (t - s)^2 \right] = \int_{t_i}^{t_{i+1}} r(j,t) \) \( t_i < j < t \)

with

\[
r(r,t) = (t-h)^2 + 3(j,t)
\]

\[
s(j,t) = v^2 \{1 + w_j \theta^2 - 3 w_j \theta + 2w_i \theta \}
\]

\[
+ v_j + v \theta^2 - 2v_j \theta + 2\theta h_i (1 - \theta) \{ -4 + 3 \theta + 2(w_i - \theta)(1-\theta) + (1-\theta)^2 \}
\]

where \( \ell_j = (t_{i+1} - T) \)

The absolute error in each interest \( I = [x_i, x_{i+1}] \). The absolute error in each

\[
|f(t) - P_i(t)| \leq 2 \beta^3 (J) \left[ R \left( \frac{t - J}{h} \right) \right]^2 dt
\]

First used to study the properties of the Kernel function \( r(f,t) \) for estimation of error, \( s(J,t) \) and evaluate the following details integrates.

\[
\int_{t_i}^{t_{i+1}} R \left( \frac{t - J}{h} \right)^2 dt = \int_{t_i}^{t_{i+1}} |r(T,x)| dJ + \int_{t_i}^{t_{i+1}} |s(T,x)| dJ
\]

To simplify the integral in [16] we begin by finding the roots of \( R(t_j - t) = 0, r(t_j, t) = 0 \) and \( s(t_j, t) = 0 \) respectively.

It is easy to see that the roots of \( r(t,j) = 0 \) in \( [0,1] \) are \( \theta_0 = 0 \), and \( \theta^* = 1 - \frac{w_i}{\ell_j} \)
where $e_i = v_i + w_i$.  \hfill (17)

Mean while roots of \( r(T, t) = 0 \) \( T_k = x - \frac{\theta h \left( \theta e_i + \left( -1 \right)^{K+1} H \right)}{1 + \theta e_i} \) \( K=1,2 \)

where \( H = \sqrt{(e_i - w_i)(1 + e_i \theta) - e_i \theta} \)
The roots of \( s(T, t) = 0 \) are \( T_3 = t_{i+1} \).
\[ T_d = t_{i-1} - \frac{2h w_i (1 - \theta) - h_i}{4 + e_i (1 - \theta)} \]

**Case I:** For \( \theta \leq \frac{w_i}{e_i} \leq 1, 0 < \theta < \theta^* \).

\[
\int_{0}^{t_i} (t_i - T)^2 dT = \left[ -\frac{h_{i+1} - \frac{T^3}{3}}{} \right]_{t_{i-1}}^{t_i} - \frac{1}{3} \left[ (t_i - t_4)^3 - (t_i - t)^3 \right] \\
= \frac{1}{3} \left[ h_i^3 \left( w_i (1 - \theta - 1) \right)^3 \right] - (1 - \theta)^3 h_i^3 \\
= \frac{1}{3} \left[ 8 \left( w_i (1 - \theta - 1) \right)^3 \right] - (1 - \theta)^3 K_i 
\]

From equation (18), we get

\[
-\frac{\theta^3 h_i^3}{3} - \frac{16}{3} h_i^3 \left( \frac{w_i (1 - \theta - 1)}{4 + e_i (1 - \theta)} \right) - \frac{6h^3}{2} \left( \frac{w_i (1 - \theta - 1)}{Q_i(x)} \right) = \ell_i (\theta, w_i, K_1, K_2) 
\]

**Case II:** For \( \theta \leq \frac{w_i}{e_i} \leq 1, \theta^* < \theta < 1, \theta^* < \theta < 1 \)

\[
\left| f(t) - P_i(t) \right| \leq \frac{1}{2} \left\| f^3(T) \right\| = \int_{t_i}^{t_{i+1}} R_i \left( t - T \right)^3 dT = \frac{1}{2} \left\| f^3(T) \right\| \ell_2 (T) \\
G_i(T) = \int_{t_i}^{t_{i+1}} r(T, t) dT + \int_{t_i}^{t_{i+1}} r(T, t) dT \\
+ \int_{t_i}^{t_{i+1}} s(T, t) dT \] 

Hence \( \left| f(t) - P_i(t) \right| \leq \left\| f^3(T) \right\| \ell_2 (v_i, w_i, \theta) \)

where \( \ell_2 (v_i, w_i, \theta) = \left[ \frac{\theta^3 h_i^3}{3} \left( \frac{\theta - H}{(1 + \theta)^3} - \frac{\theta^3 h_i^3}{3} \right) \right] \)
\[ K_i(1-\theta)^3 h_i^3 + \frac{1-\theta}{2} h_i^2 K_i + (1-\theta) h_i \] / \( Q_i(x) \)  

(21)

**Case III:** For \( \frac{w_i}{e_i} > 1, 0 < \theta < 1 \),

\[ \left| f(t_i) - p_i(t) \right| \leq \frac{1}{2} \| f^3 \| \int_{t_i}^{t_{i+1}} R_i[t-T] \] .

\[ = \frac{1}{2} \| f^3 \| \int_{t_i}^{t_{i+1}} r(T,t) dJ + \int_{t_i}^{t_{i+1}} s(T,t) dT = \frac{1}{2} \| f^3 \| e_i (v_i, w_i, \theta) \]

where \( e_i (v_i, w_i, \theta) = \frac{\theta^3 h_i^3}{3} - k_1 (1-\theta)^3 h_i^3 + k_2 (1-\theta)^2 h_i^3 (1-\theta)^2 \partial h_i \)

and \( k_1 = \left[ 1 + w_i (\theta^2 - 3\theta + 2) + v_i (1-\theta)^2 \right] \)

\( k_2 = \left[ -1 + 2 (w_i - \theta) + 3 (1-\theta) \right] \)

(22)

**Theorem 5.** The rational quintic spline interpolant defined by (1) in each sub interval.

\( I_i = [t_i, t_{i+1}] \), when \( t \in c^3(t_i, t_{i+1}) \).

The error bound is given by

\[ \left| f(t) + s(t) \right| \leq \| f'(T) \| h_i^3 e_i \]

\[ \int_{t_i}^{t_{i+1}} \left[ (T-T)^2 \right] dT \]

\[ = \| f'(T) \| \int_{t_i}^{t_{i+1}} r(T,t) dJ + \int_{t_i}^{t_{i+1}} s(T,t) dT \]

\( \theta = \| f'(T) \| h_i^3 e_i \) with

\( e_i = \max \ell_i (v_i, w_i, \theta) \)

\[ \ell_i (v_i, w_i, \theta) = \begin{cases} \ell_1 (v_i, w_i, \theta) & 0 < \theta < \theta^* \\ \ell_2 (v_i, w_i, \theta) & \theta^* \leq \theta \leq 1 \\ \ell_3 (v_i, w_i, \theta) & 0 \leq \theta \leq 1 \end{cases} \]

**5. Range restricted \( C^3 \) rational quintic function**

In this section, investigate interpolation schemes to preserve the shape of 2D Data lying above line using \( C^3 \) rational quintic function [1] Let \( \{ a = x_0, x_1, \ldots, x_n = b \} \) be the partition of the interval \( [a, b] \) and let \( f_i = f(x_i) \) \( i = 0, l, \ldots n \) and lying above straight line \( y = mx+c \). Here \( m \) is the slope of the given line and \( y \) intercept \( C \) during (1) \( f_i > v_i x_i + C \).

The rational quintic function (1) interpolating the given data set \( (x_i, f_i), i = 0, l, \ldots n \)

will be above line \( y = mx+c \) if \( s(x) > mx + C \) 

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We write
\[ s(x) = a_i(1 - \theta) + b_i\theta \quad \text{in} [x_{i-1}, x_i] \]

Here \( a_i = m_i x_i + C, b_i = m x_{i-1} + C \) and \( h_i = x_{i+1} - x_i \)

So \( \frac{P_i(\theta)}{q_i(\theta)} > a_i(1 - \theta) + b_i\theta \)

After simplification
\[ P_i(\theta) > [a_i(1 - \theta) + b_i\theta]q_i(\theta)(1 - \theta + \theta) \]
\[ \Rightarrow P_i(\theta) > a_i q_i(\theta)(1 - \theta)^2 + a_i q_i(\theta)(1 - \theta) + b_i q_i(\theta)(1 - \theta + b_iq_i(\theta)^2 \]
\[ \Rightarrow P_i(\theta) > a_i q_i(1 - \theta)^2 + (a_i + b_i) q_i(\theta) \theta(1 - \theta) + b_i q_i(\theta)^2 \]

Now \( B_0 \sigma_{i-1}(\theta) + B_i \sigma_{i+1}(\theta) + B_2 \sigma_{i-2}(\theta) + B_3 \sigma_{i+2}(\theta) + B_4 \sigma_{i+3}(\theta) + B_5 \sigma_{i+4}(\theta) \)
\[ > [a_i(1 - \theta)^2 + (a_i + b_i) \theta(1 - \theta) + b_i\theta^2]q_i(\theta) \]

where
\[ B_0 = f_i - a_i \]
\[ B_1 = f_{i+1} - b_i \]
\[ B_2 = d_i h_i + 4a_{i-1} - a_i v_i - b_i + (-3 + (v_i))f_i \]
\[ B_3 = d_{i+1} h_{i+1} + 4b_i + a_i + w_i b_i \]
\[ B_4 = \left[ D_i h_i^2 - 3d_i h_i + v_i (d_i h_i - 4f_i + 10a_i) - 2b_i - 8a_i + 10b_i - 2a_i w_i + 6f_i + 2w_i f_i \right] \]
\[ B_5 = \left[ D_{i+1} h_{i+1}^2 + w_i (-2d_{i+1} h_i - 4f_{i+1} - 8b_i) - 6d_{i+1} h_i - 2(3 - v_i) f_{i+1} - 12(a_i + b_i) - 20b_i \right] \]
\[ B_2 > 0 \quad \text{if} \quad v_i > \frac{[d_i h_i + 4a_{i-1} - b_i - 3f_i]}{f_i - a_i} \]

where \( h_i = \frac{h_i}{k_i} \)

\[ B_2 > 0 \quad \text{if} \quad w_i > \frac{d_{i+1} h_i - 4b_i + a_{i+1} f_{i+1}}{f_i - a_i} \]
\[ B_3 > 0 \quad \text{if} \quad w_i > \left[ D_{i+1} h_{i+1}^2 - 3d_{i+1} h_{i+1} - 8a_i + 6f_{i+1} + 10b_{i+1} - 2a_{i+1} w_i + 2w_i f_{i+1} \right] \]
\[ \frac{[2b_i - 10a_i + 4f_i - d_i h_i]}{2b_i - 10a_i + 4f_i - d_i h_i} \]
and \( k_i > h_i (m_i - a_i) / 3(f_i - a_i) \)

\[ B_3 > 0 \quad \text{if} \quad w_i > \left[ D_{i+1} h_{i+1}^2 + 6d_{i+1} h_{i+1} \right] \]
\[ \frac{[2(3 + v_i) f_{i+1} - 12(a_i + b_i) - 20b_i]}{d_{i+1} h_i + 4f_{i+1} + 8b_i} \]
and \( h_i (d_{i+1} - m_i) > k_i \)
C³ Rational Quintic Spline

Theorem 6. The C³ rational quintic function (1) preserve the shape of data lying above straight line \( y=mx+c \) if the shape parameters \( v_i, w_i \) satisfies the following.

**Constrains:**
\[
x_i > 0, w_i > 0
\]
\[
v_i > \max\{0, \ell_1, \ell_2\}
\]
\[
w_i > \max\{0, \ell_3, h_i\}
\]
\[
k_i > \max\{0, \ell_5, \ell_6\}
\]
i.e. \( v_i > \max\{0, \ell_1, \ell_2, \ell_3, \ell_4\} \)
\[
e_1 = -\frac{d_i h_i + 4a_i + b_i - 3f_i}{f_i - a_i}
\]
\[
e_2 = \frac{[D_i h_i^2 - 3d_i h_i - 8a_i + 6f_i + 10b_i - 2a_i w_i + 2 w_i f_i]}{2h_i - 10a_i + 4f_i - d_i h_i}
\]
\[
e_3 = \frac{d_{i+1} h_i + 4b_i + a_i + 3f_i}{(f_{i+1} - \ell_i)}
\]
\[
e_4 = \frac{D_i h_i^2 + 6d_i h_i + h_i (3 + \gamma) h_{i+1} - 12(a_i + b_i) - 20 b_i}{2a_i h_i + 4 f_{i+1} + 8 h_i}
\]
\[
e_5 = \frac{h_i (m_i - d_i)}{3(f_i - a_i)}
\]
\[
e_6 = \frac{h_i (d_{i+1} - m_i)}{3(f_{i+1} - b_i)}
\]

6. **Algorithm**

**Step I.** Input the 2D data set \((x_i, f_i)\); \(i = 0, 1, 2, \ldots, n\) lying above the straight line \( y = mx + c \).

**Step II.** Compute the values of \( h_i = x_{i+1} - x_i \), \( q_i = mx_i + C \) and \( b_i = mh_i + a_i \).

**Step III.** Compute the values of first and second order derivative \( d_i \) and \( D_i \) at knots \( x_i \) \( i = 0, 1, 2, \ldots, n \) by the arithmetic mean scheme or direct linear equation.

**Step IV.** Calculate value of the shape parameter using theorem 3.

**Step V.** Substitute the values of \( f_i, d_i, D_i \) and shape parameters in C³ rational quintic function (1) to obtain the curves lying above the straight line.
\( y = mx + c \).

6. **Result and discussion**

In the following numerical example the importance of the proposed shape preserving schemes is demonstrated through graphical results. It is conclude that the C³ continuous
quintic Hermite Polynomial (Farin) falls to preserve the inherit shape of the data and second derivative values is not possible in Malik, Zuner [7] other than A.M. but in the results obtain second derivative equal values by linear equation of second derivative $C^3$-Continuous shape. Preserving rational quintic function fills remain thing of (Farin 2005 [3, 4], Malik [7]).

7. Conclusion
The new $C^3$ rational quintic spline interpolation with (quintic/cubic) have obtained and the idea has been extended to shape preserving interpolation for positive data using the constructed rational quintic spline.

![Figure 3:](image1)

![Figure 4:](image2)

Figure 4: Rational quintic spline above straight line.
C^3 Rational Quintic Spline

REFERENCES

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