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On the Diophantine Equation $(p+12)^x + (p+2^k)^y = z^2$ where p is a Prime Number

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Abstract. In this paper, we consider the Diophantine equation $(p+12)^x + (p+2^k)^y = z^2$ where $p > 3$, p are primes and k is natural number, when x, y and z are non-negative integers. It is found that the Diophantine equation has no nonnegative integer solution.

Keywords: Diophantine equations, exponential equations, integer solution.

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

The Diophantine equation has been examined by many researchers. It is considered as one of the significant problems in the elementary number theory and the algebraic number theory. In 2014, Suvarnamani, [12] proved that the equation $p^x + (p+1)^y = z^2$ has a unique solution (p, x, y, z) = (3, 1, 0, 2) when p is an odd prime and x, y, z are non-negative integer, Additionally, in 2018, Kumar et al., [8,9] showed that the non-linear diophantine equation $p^x + (p+6)^y = z^2$ has no solution. In this year, they proof that $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$ has no solution, where x, y, z are non-negative integers, where n is a natural number. Burshtein [2,3] also proved that the Diophantine equation $2^{2x+1} + 7^y = z^2$ and showed that solutions of the Diophantine equations $p^3 + q^3 = z^2$ and $p^3 - q^3 = z^2$ when p, q are primes have no solutions. Moreover, Fernando [7] showed that the Diophantine equation $p^x + (p+8)^y = z^2$ has no solution in positive integers, when $p > 3$ and $p + 8$ are primes, Kumar et al., [10] proved that the solution of exponential Diophantine equation $p^x + (p+12)^y = z^2$, has no non-negative integer solution., when x , y, z are non-negative integers, p and p+12 are primes, let p is of the form 6n+1, where n is natural number. In 2020, Burshtein [4,5,6] showed that the diophantine equations $p^4 + q^3 = z^2$ and $p^4 - q^3 = z^2$ when p, q are distinct odd primes have no solution. In the same year, he proved that the Diophantine equation $p^x + (p+5)^y = z^2$ when p is prime where $p + 5 = 2^{2u}$ has no solution (x, y, z) in positive integers and proved that the solution to the Diophantine equation $p^x + q^y = z^3$ when ≥ 2 , q are primes, $1 \leq x, y \leq 2$ are integers. In 2021, Moonchaisook [11] proved that the non-linear Diophantine equation $p^x + (p + 4^n)^y = z^2$ has no solution, when $p > 3$, $p + 4^n$ are primes numbers, when x, y and z are non-negative integers and n is positive integer, this year Aggarwal [1] (2021) studied solutions to the exponential Diophantine equation $(2^{2m+1} - 1) + 13^n = z^2$ where m, n are whole numbers, has no solution in whole number.

Because of this open problem, the author is therefore interested in proving that the Diophantine solution $(\mathrm{p}+12)^{\mathrm{x}} + \left(\mathrm{p}+2^{\mathrm{k}}\right)^{\mathrm{y}} = \mathrm{z}^2$, has no non-negative integer solution, when x, y and z are non-negative integer, where p > 3 and p are primes, k is positive integer.

2. Basic results

Lemma 2.1. The Diophantine equation $1 + \left(\mathrm{p}+2^{\mathrm{k}}\right)^{\mathrm{y}} = \mathrm{z}^2$ has no solution in non-negative integer y, z where p > 3, p are primes and k is positive integer.

Proof: Suppose the Diophantine equation $1 + \left(\mathrm{p}+2^{\mathrm{k}}\right)^{\mathrm{y}} = \mathrm{z}^2$.

Since $\boldsymbol{p} > \mathbf{3}$ and p are primes of the form p = 4N+1 or p = 4N+3 where N is positive integer. Then $\mathrm{p} \equiv 1(\mathrm{mod}\ 4)$ or $\mathrm{p} \equiv 3\ (\mathrm{mod}\ 4)$.

Now, $1 + \left(\mathrm{p}+2^{\mathrm{k}}\right)^{\mathrm{y}}$ is an even number. Then z is even and $\mathrm{z}^2 \equiv 0(\mathrm{mod}\ 4)$.

We consider in 3 cases as follows:

Case 1: If y = 0, Then $z^2 = 2$ which is impossible.

Case 2: Suppose $\mathrm{p} \equiv 1(\mathrm{mod}\ 4)$

If k = 2n, k ≥ 1, y = 2s, s ≥ 1, then $1 + \left(\mathrm{p}+2^{\mathrm{k}}\right)^{\mathrm{y}} \equiv 2(\mathrm{mod}\ 4)$ which is a contradiction since $z^2 \equiv 0(mod\ 4)$.

If k = 2n, k ≥ 1 and y = 2s+1, s ≥ 0, then $1 + \left(\mathrm{p}+2^{\mathrm{k}}\right)^{\mathrm{y}} \equiv 2(\mathrm{mod}\ 4)$ which is a contradiction since $z^2 \equiv 0(mod\ 4)$.

If k =2n+1, k ≥ 0, y = 2s, s ≥ 1 or y = 2s+1, s ≥ 0 then $1 + \left(\mathrm{p}+2^{\mathrm{k}}\right)^{\mathrm{y}} \equiv 2(\mathrm{mod}\ 4)$ which is a contradiction since $z^2 \equiv 0(mod\ 4)$.

If k = 2n+1, k ≥ 0, y = 2s+1, s ≥ 0 then $1 + \left(\mathrm{p}+2^{\mathrm{k}}\right)^{\mathrm{y}} \equiv 2(\mathrm{mod}\ 4)$ which is a contradiction since $z^2 \equiv 0(mod\ 4)$.

Case 3: Suppose p ≡ −1 (mod 4).

If k = 2n, k ≥ 1, y = 2s, s ≥ 1 then $1 + \left(\mathrm{p}+2^{\mathrm{k}}\right)^{\mathrm{y}} \equiv 2(\mathrm{mod}\ 4)$ which is a contradiction since $z^2 \equiv 0(mod\ 4)$.

If k = 2n +1, n ≥ 0, y = 2s, s ≥ 1 then $1 + (p + 2^k)^y \equiv 2(\text{mod } 4)$ which is a contradiction since $z^2 \equiv 0(\text{mod } 4)$.

If k = 2n, n ≥ 1 or k = 2n +1, n ≥ 0, y = 2s+1, s ≥ 0

then $(p + 2^k)^{2s+1} = (p + 2^k)^{\alpha+\beta = 2s+1} = (z - 1)(z + 1)$.

Thus there exist non-negative integers and β , then $(p + 2^k)^\alpha = z + 1$ and

$(p + 2^k)^\beta = z - 1$, where $\alpha > \beta$ and α+β = 2s+1.

Therefore $(p + 2^k)^\beta \left((p + 2^k)^{\alpha-\beta} - 1\right) = 2$ .This implies that $\beta = 0$.

Then$(p + 2^k)^{2s+1} - 1 = 2$ that is $(p + 2^k)^{2s+1} = 3$ which is impossible.

Hence the Diophantine equation $1 + (p + 2^k)^y = z^2$ has no solutions.

Lemma 2.2. The Diophantine equation $(p + 12)^x + 1 = z^2$ has no solution in non-negative integer x and z where p > 3, p are primes.

Proof: Suppose the Diophantine equation $(p + 12)^x + 1 = z^2$.

For p > 3 and p are primes of the form p = 4N+1 or p = 4N+3 where N is positive integer.

then $p \equiv 1(\text{mod } 4)$ or $p \equiv 3 \ (\text{mod } 4)$ when y and z are non-negative integer.

Since $(p + 12)^x + 1 = z^2$ is even number. Thus $z^2 \equiv 0(\text{mod } 4)$.

We consider 2 cases as follows:

Case 1: If $x = 0$, Then $z^2 = 2$ which is impossible.

Case 2: If $x \geq 1$, $p > 3$ and p are primes.

Suppose $(p + 12)^x + 1 = z^2$, when x and z are non-negative integers.

Subcase 1: If x = 2m, m ≥ 1 and $p \equiv 1(\text{mod } 4)$ or $p \equiv -1 \equiv 3 \ (\text{mod } 4)$,

then $(p + 12)^x + 1 = z^2 \equiv 2$ (mod4), which is a contradiction since $z^2 \equiv 0 \ (\text{mod } 4)$.

Subcase 2: If x = 2m +1, m ≥ 0,

Suppose $(p + 12)^{2m+1} = z^2 - 1 = (z - 1)(z + 1)$.

Let $(p + 12)^\alpha = z + 1$ *and* $(p + 12)^\beta = z - 1$ or $(p + 12)^\beta + 1 = z$.

When $\alpha > \beta$ and $\alpha + \beta = 2m + 1, m > 0$.

Therefore $(p + 12)^\beta ((p + 12)^{\alpha-\beta} - 1) = 2$

If β = 0, then $(p + 12)^{2m+1} - 1 = 2$, that $(p + 12)^x = 3$ which is impossible.

If β ≥ 1, then $(p + 12)^\beta = (p + 12)^\alpha - 2$. This implies that $(p + 12)|2$ which is impossible. Hence the Diophantine equation $(p + 12)^x + 1 = z^2$ has no solution.

3. Main results

Theorem 3.1. The Diophantine equation $(p + 12)^x + (p + 2^k)^y = z^2$ has no solution in non-negative integer x, y, and z where p > 3 and p are primes and k is positive integer.

Proof: Suppose p > 3, $p + 12$ and $p + 2^k$ are primes, we have any primes of the form $p \equiv 1(\text{mod } 4)$ or $p \equiv 3 \ (\text{mod } 4)$, where y and z are non-negative integer.

Since $(p + 12)^x + (p + 2^k)^y = z^2$, z is an even number, then $z^2 \equiv 0(\text{mod } 4)$. (1)

We consider 4 cases;

Case 1: If x = 0 and y = 0 then $z^2 = 2$ which is impossible.

Case 2: If x = 0, y ≥ 1 then $1 + (p + 2^k)^y = z^2$ has no solution by Lemma 2.1.

Case 3: If y = 0, x ≥ 1 then $(p+12)^x + 1 = z^2$ has no solution by Lemma 2.2.
Case 4: If x ≥ 1, y ≥ 1 then $(p+12)^x + (p+2^k)^y = z^2$ has no solutions.
We consider in 4 subcases as follows;

Subcase 1: If x = 2m, m ≥ 1 and $y = 2s, s \geq 1$,then

If $p \equiv 1 (\mathrm{mod}\ 4)$ then $(p+12)^x + (p+2^k)^y = z^2 \equiv 2 (\mathrm{mod}\ 4)$. (2)

Equation (2) contradicts equation (1).
Hence the Diophantine equation $(p+12)^x + (p+2^k)^y = z^2$ has no solution.

If $p \equiv -1 \equiv 3\ (mod\ 4)$ then $(p+12)^x + (p+2^k)^y = z^2 \equiv 2 (\mathrm{mod}\ 4)$ (3)

Equation (3) contradicts equation (1).
Hence the Diophantine equation $(p+12)^x + (p+2^k)^y = z^2$ has no solution.

Subcase 2: If x = 2m+1 and y = 2s+1, where m ≥ 0 , s ≥ 0 and m, s are positive integers.

If $p \equiv 1 (\mathrm{mod}\ 4)$, so that $(p+12)^{2m+1} + (p+2^k)^{2s+1} \equiv 2\ (\mathrm{mod}\ 4)$. (4)

The equation (4) contradicts equation (1).
Hence the Diophantine equation $(p+12)^x + (p+2^k)^y = z^2$ has no solution.

If $p \equiv -1 \equiv 3\ (mod\ 4)$ then $(p+12)^{2m+1} + (p+2^k)^{2s+1} \equiv 2 (\mathrm{mod}\ 4)$ (5)

The equation (5) contradicts equation (1).
Hence the Diophantine equation $(p+12)^x + (p+2^k)^y = z^2$ has no solution.

Subcase 3: If $x = 2m+1$, $m \geq 0, y = 2s, s \geq 1$,
Since $(p+12)^x + (p+2^k)^{2s} = z^2$, then $(p+12)^x = z^2 - (p+2^k)^{2s}$,
then $(p+12)^{2m+1} = (z+(p+2^k)^s)(z-(p+2^k)^s)$. Thus there exist non-negative integers α and β then $(p+12)^\alpha = z + (p+2^k)^s$ and $(p+12)^\beta + (p+2^k)^s = z$ (6)
when α > β and $\alpha + \beta = 2m+1$. Therefore $(p+12)^\beta((p+12)^{\alpha-\beta} - 1) = 2(p+2^k)^s$

If β = 0, then $(p+12)^{2m+1} - 1 = 2(p+2^k)^s \equiv 2(mod\ 4)$ (7)

If $p \equiv 1 (\mathrm{mod}\ 4)$ then $(p+12)^{2m+1} - 1 \equiv 0\ (mod\ 4)$. (8)

Equation (8) contradicts equation (7).
Hence the Diophantine equation$(p+12)^x + (p+2^k)^y = z^2$ has no solution.

If $p \equiv -1\ (\mathrm{mod}\ 4)$ then $(p+12)^{2m+1} + (p+2^k)^{2s} = z^2 \equiv 2 (\mathrm{mod}\ 4)$ (9)

Equation (9) contradicts equation (1).
Hence the Diophantine equation$(p+12)^x + (p+2^k)^y = z^2$ has no solution.

If β ≥ 1, from (6) then $(p+12)^\alpha = (p+12)^\beta + 2(p+2^k)^s$ where $\alpha > \beta$ and α+β = $2m+1$ then imply that $(p+12) \mid 2(p+2^k)^s$ which is impossible since p+12 is an odd number but $2(p+2^k)^s$ is an even number.
Hence exponential Diophantine equation $(p+12)^x + (p+2^k)^y = z^2$, has no solution.

Subcase 4: If x = 2m, $m \geq 1$, $y = 2s+1, s \geq 0$,

then$(\mathrm{p}+12)^{\mathrm{x}} + (\mathrm{p}+2^{\mathrm{k}})^{2\mathrm{s}+1} = (\mathrm{z}+(\mathrm{p}+12)^{\mathrm{m}})(\mathrm{z}-(\mathrm{p}+12)^{\mathrm{m}})$.
where $\alpha > \beta$ and $\alpha+\beta = 2s+1$.
then $(\mathrm{p}+2^{\mathrm{k}})^{\alpha} = \mathrm{z}+(\mathrm{p}+12)^{\mathrm{m}}$ and $(\mathrm{p}+2^{\mathrm{k}})^{\beta} + (\mathrm{p}+12)^{\mathrm{m}} = \mathrm{z}$ (10)
From the equation (10) then $(\mathrm{p}+2^{\mathrm{k}})^{\alpha} = (\mathrm{p}+2^{\mathrm{k}})^{\beta} + 2(\mathrm{p}+12)^{\mathrm{m}}$ This implies that $(\mathrm{p}+2^{\mathrm{k}}) | 2(\mathrm{p}+12)^{\mathrm{m}}$ which is impossible since $(\mathrm{p}+2^{\mathrm{k}})$ is an odd number and $2(\mathrm{p}+12)^{m}$ is an even number.
Hence the Diophantine equation $(\mathrm{p}+12)^{\mathrm{x}} + (\mathrm{p}+2^{\mathrm{k}})^{\mathrm{y}} = \mathrm{z}^2$ has no solution.

Corollary 3.1.1. The Diophantine equation $(\mathrm{p}+12)^{\mathrm{x}} + (\mathrm{p}+2^{\mathrm{k}})^{\mathrm{y}} = \mathrm{m}^4$ has no solution in non-negative integer m, x and y where k is positive integer, p > 3 and p are primes.
Proof: Suppose $\mathrm{m}^2 = \mathrm{z}$, then $(\mathrm{p}+12)^{\mathrm{x}} + (\mathrm{p}+2^{\mathrm{k}})^{\mathrm{y}} = \mathrm{m}^4$, which has no non-negative integer solution by Theorem 3.1.

Corollary 3.1.2. The Diophantine equation $(\mathrm{p}+12)^{\mathrm{x}} + (\mathrm{p}+2^{\mathrm{k}})^{\mathrm{y}} = \mathrm{m}^{2\mathrm{n}+2}$ has no solution in non-negative integer m, x and y where k is positive integer, p > 3 and p are primes.
Proof: Suppose $\mathrm{m}^{\mathrm{n}+1} = z$ then $(\mathrm{p}+12)^{\mathrm{x}} + (\mathrm{p}+2^{\mathrm{k}})^{\mathrm{y}} = \mathrm{w}^{2\mathrm{n}+2} = \mathrm{z}^2$ has no non-negative integer solution by Theorem 3.1.

4. Conclusion

In this paper, we proved that the Diophantine equation $(\mathrm{p}+12)^{\mathrm{x}} + (\mathrm{p}+2^{\mathrm{k}})^{\mathrm{y}} = \mathrm{z}^2$ has no solution where x, y and z are non-negative and n is positive integer. There are infinitely many the exponential Diophantine equation that has no solution which need to be proved.

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