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Strongly Proximinal Subspaces in Orlicz Function Space

G. Ghawadrah *

Department of Mathematics, Faculty of Science, An-Najah National University
Nablus, Palestine. E-mail: g.ghawadrah@najah.edu

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Abstract. In this paper, we prove that if Y is a separable proximinal subspace of X, then Y is strongly proximinal in X if and only if $L^{\phi}(\mu, Y)$ is strongly proximinal in $L^{\phi}(\mu, X)$. where $L^{\phi}(\mu, X)$ is an Orlicz function space with Luxemburg norm.

Keywords: Strong proximinality, Orlicz function space

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1. Introduction

Let $(X, \|.\|_X)$ be a normed linear space and G be a subset of X. For $x \in X$, let $d(x, G) = \inf\{\|x-g\|_X : g \in G\}$ and let $P_G(x) = \{g \in G : \|x-g\|_X = d(x, G)\}$. If G is a subspace of X, an element $g_0 \in G$ is called a best approximant of x in G if $g_0 \in P_G(x)$. Moreover, If for each $x \in X$, $P_G(x) \neq \emptyset$, then G is said to be proximinal in X, for more see [5] and [6]. We recall the following definition of stronger version of approximation:

Definition 1.1. [4] A closed convex subset C of a Banach space X is said to be strongly proximinal if it is proximinal and for a given $x \in X \setminus C$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $P_C(x, \delta) \subseteq P_C(x) + \varepsilon B_X$, where $P_C(x, \delta) = \{z \in C : \|x - z\|_X \leq d(x, C) + \delta\}$.

From the definition of strong proximinality, it is clear that if Y is a strongly proximinal subspace of X, then the metric projection $P_Y : X \to 2^Y$ is upper Hausdorff semi-continuous, abbreviated uHsc, for more see [3].

Let ϕ be an Orlicz function on $[0, \infty)$ (i.e. a continuous, strictly increasing, convex function satisfying $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$). Let (Ω, Σ, μ) be a measure space. An Orlicz space $L^{\phi}(\mu)$ is a space of all measurable functions $f : \Omega \longrightarrow \mathbb{R}$ such that $\int_{\Omega} \phi(c^{-1}|f(t)|)\, d\mu(t) < \infty$, for some $c > 0$, with norm

$$\|f\|_{\phi} = \inf\left\{c > 0 : \int_{\Omega} \phi(c^{-1}|f(t)|)\, d\mu(t) \leq 1\right\}.$$

Let M^{ϕ} be a subspace of $L^{\phi}(\mu)$ such that for all $c > 0$, $\int_{\Omega} \phi(c^{-1}|f(t)|)\, d\mu(t) < \infty$.

For a real Banach space $(X, \|.\|_X)$, The Orlicz space $L^{\phi}(\mu, X)$ is a space of all strongly measurable functions $f : \Omega \longrightarrow X$ such that $\int_{\Omega} \phi(c^{-1}\|f(t)\|_X)\, d\mu(t) < \infty$, for some $c > 0$. Define a Luxemburg norm on $L^{\phi}(\mu, X)$ by

$$\|f\|_\phi = \inf\left\{c > 0 : \int_\Omega \phi(c^{-1}\|f(t)\|_X)\, d\mu(t) \le 1\right\},$$

the subspace $M^\phi(X)$ contains all strongly measurable functions $f:\Omega \longrightarrow X$ such that for all $c > 0$, $\int_\Omega \phi(c^{-1}\|f(t)\|_X)\, d\mu(t) < \infty$. The function ϕ is said to satisfy Δ_2 – condition, denoted $\phi \in \Delta_2$ if $\phi(2t) \le K\phi(t), t \ge t_0 \ge 0$, for some absolute constant $K > 0$, also we say ϕ is Δ_2 - regular if $\phi \in \Delta_2$. It is known that if ϕ is Δ_2 – regular, then $M^\phi(X) = L^\phi(\mu, X)$, $M^\phi = L^\phi(\mu)$, for more about Orlicz function spaces see [2]. There are many results about best approximation in Orlicz function space, reader is referred to [1,7,8,9,10]. In [3], Paul investigated the strong proximinality and ball proximinality in $L^p(\mu, X), 1 \le p \le \infty$.

In this paper, we will prove that if Y is a separable proximinal subspace of X, then Y is strongly proximinal in X if and only if $L^\phi(\mu, Y)$ is strongly proximinal in $L^\phi(\mu, X)$.

2. Main results

Throughout this paper we suppose μ is a Lebesgue measure on $\Omega = [0,1]$, ϕ is Δ_2-regular ($\phi \in \Delta_2$), $\phi(1) = 1$ and X is a real Banach space.

Theorem 2.1. Let Y be a separable proximinal subspace of X such that P_Y is uHsc. Then for $f \in L^\phi(\mu, Y)$, $g \in L^\phi(\mu, X)$:

$$d\left(f, P_{L^\phi(\mu,Y)}(g)\right) = \|d(f(.), P_Y(g(.)))\|_\phi$$

Proof: From [1, Corollary 2.1] it implies that $h \in P_{L^\phi(\mu,Y)}(g)$ if and only if $h(t) \in P_Y(g(t))$ a.e.
Thus, for every $h \in P_{L^\phi(\mu,Y)}(g)$, $\|f(t) - h(t)\| \ge d(f(t), P_Y(g(t)))$ a.e. Since ϕ is strictly increasing, then for every positive constant c we have

$$\phi(c^{-1}\|f(t) - h(t)\|) \ge \phi\left(c^{-1} d(f(t), P_Y(g(t)))\right)$$

Hence, for every $h \in P_{L^\phi(\mu,Y)}(g)$, $\|f - h\|_\phi \ge \|d(f(.), P_Y(g(.)))\|_\phi$.
Therefore, $d\left(f, P_{L^\phi(\mu,Y)}(g)\right) = inf_{h \in P_{L^\phi(\mu,Y)}(g)}\|f - h\|_\phi \ge \|d(f(.), P_Y(g(.)))\|_\phi$.
From [3, Lemma 3.3] there is a sequence of measurable selections $\{h_n\}_{n=1}^\infty$ where for all t, $h_n(t) \in P_{P_Y(g(t))}\left(f(t), \frac{1}{n}\right)$, which leads to the inequality:

$$d(f(t), P_Y(g(t))) \le \|f(t) - h_n(t)\|_X \le d(f(t), P_Y(g(t))) + \frac{1}{n}$$

Hence, $\lim_{n\to\infty}\|f(t) - h_n(t)\|_X = d(f(t), P_Y(g(t)))$.
Let $R_n(t) = \|f(t) - h_n(t)\|_X$ and $K(t) = d(f(t), P_Y(g(t)))$, $t \in [0,1]$, then $K, R_n \in L^\phi(\mu)(= M^\phi)$.
Hence, for any fixed $c > 0$, $\lim_{n\to\infty}\phi\left(\frac{R_n(t) - K(t)}{c}\right) = 0$ and $\phi\left(\frac{R_n(t) - K(t)}{c}\right) \le \phi(\frac{1}{c}), t \in [0,1]$.

Therefore, $\lim_{n\to\infty}\int_0^1 \phi\left(\frac{R_n(t) - K(t)}{c}\right) d\mu(t) = 0$ by dominated convergence theorem, so [Lemma 1, p. 157] in [2] implies $\lim_{n\to\infty}\|R_n - K\|_\phi = 0$ in $L^\phi(\mu)$. Hence,

$\lim_{n\to\infty}\|R_n\|_\phi = \|K\|_\phi$ in $L^\phi(\mu)$. Thus, $\lim_{n\to\infty}\|f-h_n\|_\phi = \|d(f(.),P_Y(g(.)))\|_\phi$ in $L^\phi(\mu,X)$.

For all $\in[0,1]$, $h_n(t)\in P_Y\big(g(t)\big)$, so [1, Corollary 2.1] implies $h_n\in P_{L^\phi(\mu,Y)}(g)$. Therefore, $d\Big(f,P_{L^\phi(\mu,Y)}(g)\Big)\le \lim_{n\to\infty}\|f-h_n\|_\phi = \|d(f(.),P_Y(g(.))\,)\|_\phi$.

Thus, $d\Big(f,P_{L^\phi(\mu,Y)}(g)\Big) = \|d(f(.),P_Y(g(.))\,)\|_\phi$. ∎

Theorem 2.2. Let Y be a separable proximinal subspace of X. Y is strongly proximinal in X if and only if $L^\phi(\mu,Y)$ is strongly proximinal in $L^\phi(\mu,X)$.

Proof: ($\Longrightarrow$) Let Y be strongly proximinal in X and $L^\phi(\mu,Y)$ be not strongly proximinal in $L^\phi(\mu,X)$.

Hence, $\exists f\in L^\phi(\mu,X)\setminus L^\phi(\mu,Y)$ and $\exists\varepsilon>0$ such that $\forall n\in\mathbb{N},\exists g_n\in P_{L^\phi(\mu,Y)}(f,\frac{1}{n})$ and $d\Big(g_n,P_{L^\phi(\mu,Y)}(f)\Big)\ge\varepsilon$.

Thus, $0<d\big(f,L^\phi(\mu,Y)\,\big)\le\|f-g_n\|_\phi\le d\big(f,L^\phi(\mu,Y)\,\big)+\frac{1}{n}$ (1)

so it is clear that $\lim_{n\to\infty}\|f-g_n\|_\phi = d\big(f,L^\phi(\mu,Y)\,\big) = \|d(f(.),Y)\|_\phi$.

Since for all $t\in[0,1]$, $d(f(t),Y)\le\|f(t)\|_X$, then inequality (1) implies that $\frac{d(f(t),Y)}{\beta_n}\le\frac{\|f(t)\|_X}{\alpha}$ where $\alpha = d\big(f,L^\phi(\mu,Y)\,\big)$, and $\beta_n = \|f-g_n\|_\phi$. Since ϕ is increasing and Δ_2- regular, then $\phi\left(\frac{d(f(t),Y)}{\beta_n}\right)\le\phi\left(\frac{\|f(t)\|_X}{\alpha}\right)$, $\int_0^1\phi\left(\frac{\|f(t)\|_X}{\alpha}\right)d\mu\,(t)<\infty$, (i.e. $\phi\left(\frac{\|f(.)\|_X}{\alpha}\right)\in L^1(\mu)$) and $\lim_{n\to\infty}\phi\left(\frac{d(f(t),Y)}{\beta_n}\right)=\phi\left(\frac{d(f(t),Y)}{\alpha}\right)$ for all $t\in[0,1]$. Therefore, dominated convergence theorem in $L^1(\mu)$ implies that

$$\lim_{n\to\infty}\int_0^1\phi\left(\frac{d(f(t),Y)}{\beta_n}\right)d\mu(t) = \int_0^1\phi\left(\frac{d(f(t),Y)}{\alpha}\right)d\mu(t).$$

Lemma 2.2 and Remark 2.1in [1] (or by [Proposition 6, p. 77] in [2]) imply that $\int_0^1\phi\left(\frac{d(f(t),Y)}{\alpha}\right)d\mu(t)=1$ and $\forall n, \int_0^1\phi\left(\frac{\|f(t)-g_n(t)\|_X}{\beta_n}\right)d\mu(t)=1$. Hence, we have the following

$$\lim_{n\to\infty}\int_0^1\phi\left(\frac{\|f(t)-g_n(t)\|_X}{\beta_n}\right)d\mu(t) = \int_0^1\phi\left(\frac{d(f(t),Y)}{\alpha}\right)d\mu(t),$$

$$\Longrightarrow \lim_{n\to\infty}\int_0^1\left(\phi\left(\frac{\|f(t)-g_n(t)\|_X}{\beta_n}\right)-\phi\left(\frac{d(f(t),Y)}{\beta_n}\right)\right)d\mu(t)=0.$$

$$\Longrightarrow \lim_{n\to\infty}\int_0^1\left|\phi\left(\frac{\|f(t)-g_n(t)\|_X}{\beta_n}\right)-\phi\left(\frac{d(f(t),Y)}{\beta_n}\right)\right|d\mu(t)=0.$$

Then there exists a subsequence $\left\{\phi\left(\frac{\left\|f(t)-g_{n_i}(t)\right\|_X}{\beta_{n_i}}\right)-\phi\left(\frac{d(f(t),Y)}{\beta_{n_i}}\right)\right\}$ converges to 0 a.e. and hence $\lim_{i\to\infty}\phi\left(\frac{\left\|f(t)-g_{n_i}(t)\right\|_X}{\beta_{n_i}}\right)=\phi\left(\frac{d(f(t),Y)}{\alpha}\right)$. Since ϕ is continuous and strictly increasing, then ϕ^{-1} is a continuous function , so we get that $\lim_{i\to\infty}\frac{\left\|f(t)-g_{n_i}(t)\right\|_X}{\beta_{n_i}} = \frac{d(f(t),Y)}{\alpha}$ a.e.

Hence, $\lim_{i\to\infty}\left\|f(t)-g_{n_i}(t)\right\|_X = d(f(t),Y)$ a.e. because $\lim_{i\to\infty}\beta_{n_i}=\alpha$. Now, since Y is strongly proximinal, then we have that $\lim_{i\to\infty} d\left(g_{n_i}(t),P_Y(f(t))\right)=0$.

Since $\forall c>0$, $\phi\left(\frac{d\left(g_{n_i}(t),P_Y(f(t))\right)}{c}\right)\le\phi\left(\frac{2\|f(t)\|_X}{c}\right)$ a.e. and $\int_0^1\phi\left(\frac{2\|f(t)\|_X}{\alpha}\right)d\mu\,(t)<\infty$, (i.e. $\phi\left(\frac{2\|f(.)\|_X}{\alpha}\right)\in L^1(\mu)$), then dominated convergence theorem in $L^1(\mu)$ implies that $\lim_{i\to\infty}\int_0^1\phi\left(\frac{d\left(g_{n_i}(t),P_Y(f(t))\right)}{c}\right)d\mu\,(t)=0$. Therefore, by [Lemma 1, p.157] in [2] we have that $\lim_{i\to\infty} d\left(g_{n_i},P_{L^{\phi}(\mu,Y)}(f)\right)=\lim_{i\to\infty}\left\|d\left(g_{n_i}(t),P_Y(f(t))\right)\right\|_{\phi}=0$, which contradicts our assumption on $\{g_n\}_{n=1}^{\infty}$.

($\Leftarrow$)Let $L^{\phi}(\mu,Y)$ be strongly proximinal in $L^{\phi}(\mu,X)$ and Y be not strongly proximinal in X.

Hence, $\exists x\in X\ Y$ and $\exists\varepsilon>0$ such that $\forall\delta>0,\exists y_\delta\in P_Y(x,\delta)$ and $d\left(y_\delta,P_Y(x)\right)>\varepsilon$. Consider $f(t)=x$ and $g_\delta(t)=y_\delta$ for all $t\in[0,1]$, so $f\in L^{\phi}(\mu,X)$ and $\forall\delta, g_\delta\in L^{\phi}(\mu,Y)$. Since $d\left(f,L^{\phi}(\mu,Y)\right)=d(x,Y)$ and $d\left(g_\delta,P_{L^{\phi}(\mu,Y)}(f)\right)=d\left(y_\delta,P_Y(x)\right)$ by Theorem 2.1, then $g_\delta\in P_{L^{\phi}(\mu,Y)}(f,\delta)$ and $d\left(g_\delta,P_{L^{\phi}(\mu,Y)}(f)\right)>\varepsilon$ and hence $L^{\phi}(\mu,Y)$ is not strongly proximinal, which is a contradiction. ∎

3. Conclusion

We conclude that if Y is a separable proximinal subspace of X, then Y is strongly proximinal in X if and only if $L^{\phi}(\mu,Y)$ is strongly proximinal in $L^{\phi}(\mu,X)$.

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