Strongly Proximinal Subspaces in Orlicz Function Space

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Abstract. In this paper, we prove that if $L^\Phi(\mu, Y)$ is a separable proximinal subspace of $L^\Phi(\mu, \mathbb{X})$, then $L^\Phi(\mu, Y)$ is strongly proximinal in $L^\Phi(\mu, \mathbb{X})$ if and only if $L^\Phi(\mu, Y)$ is strongly proximinal in $L^\Phi(\mu, \mathbb{X})$. Where $L^\Phi(\mu, \mathbb{X})$ is an Orlicz function space with Luxemburg norm.

Keywords: Strong proximinality, Orlicz function space

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1. Introduction

Let $(\Omega, \Sigma, \mu)$ be a measure space. An Orlicz space $L^\Phi(\mu)$ is a space of all strongly measurable functions $f: \Omega \to \mathbb{X}$ such that

$$\int_\Omega \Phi(c^{-1}|f(t)|) \, d\mu(t) < \infty,$$

for some $c > 0$, with norm

$$\|f\|_\Phi = \inf\left\{c > 0 : \int_\Omega \Phi(c^{-1}|f(t)|) \, d\mu(t) \leq 1\right\}.$$

Let $M^\Phi$ be a subspace of $L^\Phi(\mu)$ such that for all $c > 0$, $\int_\Omega \Phi(c^{-1}|f(t)|) \, d\mu(t) < \infty$. For a real Banach space $(X, \|\cdot\|_X)$, the Orlicz space $L^\Phi(\mu, X)$ is a space of all strongly measurable functions $f: \Omega \to X$ such that $\int_\Omega \Phi(c^{-1}\|f(t)\|_X) \, d\mu(t) < \infty$, for some $c > 0$. Define a Luxemburg norm on $L^\Phi(\mu, X)$ by...
Therefore, from [3, Lemma 3.3] there is a sequence of measurable selections

\[ \text{let } \text{Theorem 2.1. } \]

Hence, for every positive constant \( K > 0 \), also we say \( \Phi \) is \( \Delta_2 \)-regular if \( \phi \in \Delta_2 \). It is known that if \( \phi \) is \( \Delta_2 \)-regular, then \( M^\phi(X) = L^\phi(\mu, X) \), \( M^\phi = L^\phi(\mu) \), for more about Orlicz function spaces see [2].

There are many results about best approximation in Orlicz function space, reader is referred to [1,7,8,9,10]. In [3], Paul investigated the strong proximinality and ball proximinality in \( L^\phi(\mu, X) \), \( 1 \leq p \leq \infty \).

In this paper, we will prove that if \( Y \) is a separable proximal subspace of \( X \), then \( Y \) is strongly proximal in \( X \) if and only if \( L^\phi(\mu, Y) \) is strongly proximal in \( L^\phi(\mu, X) \).

### 2. Main results

Throughout this paper we suppose \( \mu \) is a Lebesgue measure on \( \Omega = [0,1] \), \( \Phi \) is \( \Delta_2 \)-regular \( (\phi \in \Delta_2) \), \( \Phi(1) = 1 \) and \( X \) is a real Banach space.

**Theorem 2.1.** Let \( Y \) be a separable proximal subspace of \( X \) such that \( P_Y \) is uHsc. Then for \( f \in L^\phi(\mu, Y) \), \( g \in L^\phi(\mu, X) \):

\[ d(f, P_{L^\phi(\mu,Y)}(g)) = \|d(f(\cdot), P_Y(g(\cdot)))\|_\Phi \]

**Proof:** From [1, Corollary 2.1] it implies that \( h \in P_{L^\phi(\mu,Y)}(g) \) if and only if \( h(t) \in P_Y(g(t)) \) a.e.

Thus, for every \( h \in P_{L^\phi(\mu,Y)}(g) \), \( \|f(t) - h(t)\| \geq d(f(t), P_Y(g(t))) \) a.e. Since \( \Phi \) is strictly increasing, then for every positive constant \( c \) we have

\[ \phi(c^{-1}\|f(t) - h(t)\|) \geq \phi(c^{-1}d(f(t), P_Y(g(t)))) \]

Hence, for every \( h \in P_{L^\phi(\mu,Y)}(g) \), \( \|f - h\|_\Phi \geq \|d(f(\cdot), P_Y(g(\cdot)))\|_\Phi \).

Therefore, \( d(f, P_{L^\phi(\mu,Y)}(g)) = \inf_{h \in P_{L^\phi(\mu,Y)}(g)} \|f - h\|_\Phi \geq \|d(f(\cdot), P_Y(g(\cdot)))\|_\Phi \).

From [3, Lemma 3.3] there is a sequence of measurable selections \( \{h_n\}_{n=1}^\infty \) where for all \( t, h_n(t) \in P_{P_Y(g(t))} \left( f(t), \frac{1}{n} \right) \), which leads to the inequality:

\[ d(f(t), P_Y(g(t))) \leq \|f(t) - h_n(t)\|_X \leq d(f(t), P_Y(g(t))) + \frac{1}{n} \]

Hence, \( \lim_{n \to \infty} \|f(t) - h_n(t)\|_X = d(f(t), P_Y(g(t))) \).

Let \( R_n(t) = \|f(t) - h_n(t)\|_X \) and \( K_n(t) = d(f(t), P_Y(g(t))) \). \( t \in [0,1] \), then \( K_n, R_n \in \text{L}^\phi(\mu)(= M^\phi) \).

Hence, for any fixed \( c > 0 \), \( \lim_{n \to \infty} \phi \left( \frac{R_n(t) - K(t)}{c} \right) = 0 \) and \( \phi \left( \frac{R_n(t) - K(t)}{c} \right) \leq \phi \left( \frac{1}{c} \right) \), \( t \in [0,1] \).

Therefore, \( \lim_{n \to \infty} \int_0^1 \phi \left( \frac{R_n(t) - K(t)}{c} \right) d\mu(t) = 0 \) by dominated convergence theorem, so [Lemma 1, p. 157] in [2] implies \( \lim_{n \to \infty} \|R_n - K\|_\Phi = 0 \) in \( L^\phi(\mu) \). Hence,
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\[ \lim_{n \to \infty} \| R_n \|_\phi = \| K \|_\phi \text{ in } L^\Phi(\mu). \] Thus, \[ \lim_{n \to \infty} \| f - h_n \|_\phi = \| d(f(\cdot), P_Y(g(\cdot))) \|_\phi \text{ in } L^\Phi(\mu, X). \]

For all \[ [0, 1], h_n(t) \in P_Y(g(t)), \] so [1, Corollary 2.1] implies \( h_n \in L^\Phi(\mu, Y). \)

Therefore, \[ d(f, P_{L^\Phi(\mu, Y)}(g)) \leq \lim_{n \to \infty} \| f - h_n \|_\phi = \| d(f(\cdot), P_Y(g(\cdot))) \|_\phi. \]

Thus, \[ d(f, P_{L^\Phi(\mu, Y)}(g)) = \| d(f(\cdot), P_Y(g(\cdot))) \|_\phi. \]

\[ \square \]

**Theorem 2.2.** Let \( Y \) be a separable proximinal subspace of \( X \). \( Y \) is strongly proximinal in \( X \) if and only if \( L^\Phi(\mu, Y) \) is strongly proximinal in \( L^\Phi(\mu, X) \).

**Proof:** (\( \Rightarrow \)) Let \( Y \) be strongly proximinal in \( X \) and \( L^\Phi(\mu, Y) \) be not strongly proximinal in \( L^\Phi(\mu, X) \).

Hence, \( \exists f \in L^\Phi(\mu, X) \setminus L^\Phi(\mu, Y) \) and \( \exists \varepsilon > 0 \) such that \( \forall n \in \mathbb{N}, \exists g_n \in P_{L^\Phi(\mu, Y)}(f, \frac{1}{n}) \) and \( d(g_n, P_{L^\Phi(\mu, Y)}(f)) \geq \varepsilon. \)

Thus, \( 0 < d(f, L^\Phi(\mu, Y)) \leq \| f - g_n \|_\phi \leq d(f, L^\Phi(\mu, Y)) + \frac{1}{n} \) for all \( t \in [0, 1], \) \( d(f(t), Y) \leq \| f(t) \|_X, \) then inequality (1) implies that

\[ \frac{d(f(t), Y)}{\beta_n} \leq \frac{\| f(t) \|_X}{\alpha} \text{ where } \alpha = d(f, L^\Phi(\mu, Y)), \text{ and } \beta_n = \| f - g_n \|_\phi. \]

Since \( \phi \) is increasing and \( \Delta_2 \)-regular, then \( \phi \left( \frac{d(f(t), Y)}{\beta_n} \right) \leq \phi \left( \frac{\| f(t) \|_X}{\alpha} \right), \) \( \int_0^1 \phi \left( \frac{\| f(t) \|_X}{\alpha} \right) d\mu(t) < \infty, \)

(i.e. \( \phi \left( \frac{\| f(t) \|_X}{\alpha} \right) \in L^1(\mu) \) and \( \lim_{n \to \infty} \phi \left( \frac{d(f(t), Y)}{\beta_n} \right) \phi \left( \frac{d(f(t), Y)}{\beta_n} \right) \) for all \( t \in [0, 1]. \)

Therefore, dominated convergence theorem in \( L^1(\mu) \) implies that

\[ \lim_{n \to \infty} \int_0^1 \phi \left( \frac{d(f(t), Y)}{\beta_n} \right) d\mu(t) = \int_0^1 \phi \left( \frac{d(f(t), Y)}{\alpha} \right) d\mu(t). \]

Lemma 2.2 and Remark 2.1 in [1] (or by [Proposition 6, p. 77] in [2]) imply that

\[ \int_0^1 \phi \left( \frac{\| f(t) - g_n(t) \|_X}{\beta_n} \right) d\mu(t) = 1 \text{ and } \forall n, \int_0^1 \phi \left( \frac{\| f(t) - g_n(t) \|_X}{\beta_n} \right) d\mu(t) = 1. \]

Hence, we have the following

\[ \lim_{n \to \infty} \int_0^1 \phi \left( \frac{\| f(t) - g_n(t) \|_X}{\beta_n} \right) d\mu(t) = \int_0^1 \phi \left( \frac{d(f(t), Y)}{\beta_n} \right) d\mu(t), \]

\[ \implies \lim_{n \to \infty} \int_0^1 \phi \left( \frac{\| f(t) - g_n(t) \|_X}{\beta_n} \right) - \phi \left( \frac{d(f(t), Y)}{\beta_n} \right) d\mu(t) = 0. \]

\[ \implies \lim_{n \to \infty} \int_0^1 \phi \left( \frac{\| f(t) - g_n(t) \|_X}{\beta_n} \right) - \phi \left( \frac{d(f(t), Y)}{\beta_n} \right) d\mu(t) = 0. \]

Then there exists a subsequence \( \left\{ \phi \left( \frac{\| f(t) - g_{n_i}(t) \|_X}{\beta_{n_i}} \right) - \phi \left( \frac{d(f(t), Y)}{\beta_{n_i}} \right) \right\} \) converges to 0 a.e.

and hence

\[ \lim_{i \to \infty} \phi \left( \frac{\| f(t) - g_{n_i}(t) \|_X}{\beta_{n_i}} \right) = \phi \left( \frac{d(f(t), Y)}{\alpha} \right). \]

Since \( \phi \) is continuous and strictly increasing, then \( \phi^{-1} \) is a continuous function, so we get that

\[ \lim_{i \to \infty} \frac{d(f(t), Y)}{\beta_{n_i}} = \frac{d(f(t), Y)}{\alpha} \text{ a.e.} \]

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Hence, $\lim_{t \to \infty} ||f(t) - g_{n_i}(t)||_x = d(f(t), Y)$ a.e. because $\lim \beta_{n_i} = \alpha$. Now, since $Y$ is strongly proximinal, then we have that $\lim_{t \to \infty} d \left( g_{n_i}(t), P_Y(f(t)) \right) = 0$.

Since $\forall c > 0$, $\phi(\frac{d(\|g_{n_i}(t), P_Y(f(t))\|)}{c}) \leq \phi(\frac{2\|f(t)\|_x}{c})$ a.e. and $\int_0^1 \phi \left( \frac{2\|f(t)\|_x}{\alpha} \right) d\mu(t) < \infty$, (i.e. $\phi(\frac{2\|f(t)\|_x}{\alpha}) \in L^1(\mu)$), then dominated convergence theorem in $L^1(\mu)$ implies that $\lim_{t \to \infty} \int_0^1 \phi \left( \frac{d(\|g_{n_i}(t), P_Y(f(t))\|)}{c} \right) d\mu(t) = 0$. Therefore, by [Lemma 1, p.157] in [2] we have that $\lim_{t \to \infty} d \left( g_{n_i}, P_{L_\phi(\mu,Y)}(f) \right) = \lim_{t \to \infty} \| d \left( g_{n_i}(t), P_Y(f(t)) \right) \|_\phi = 0$, which contradicts our assumption on $\{g_{n_i}\}_{i=1}^\infty$.

(⇐) Let $L_\phi(\mu, Y)$ be strongly proximinal in $L_\phi(\mu, X)$ and $Y$ be not strongly proximinal in $X$.

Hence, $\exists x \in X \setminus Y$ and $\exists \varepsilon > 0$ such that $\forall \theta > 0$, $\exists \gamma_{\theta} \in P_Y(x, \delta)$ and $d(\gamma_{\theta}, P_Y(x)) > \varepsilon$.

Consider $f(t) = x$ and $g_{\delta}(t) = \gamma_{\theta}$ for all $t \in [0,1]$, so $f \in L_\phi(\mu, X)$ and $\forall \delta, \gamma_{\theta} \in L_\phi(\mu, Y)$. Since $d(f, L_\phi(\mu, Y)) = d(x, Y)$ and $d(g_{\delta}, P_{L_\phi(\mu,Y)}(f)) = d(\gamma_{\theta}, P_Y(x))$ by Theorem 2.1, then $g_{\delta} \in P_{L_\phi(\mu,Y)}(f, \delta)$ and $d \left( g_{\delta}, P_{L_\phi(\mu,Y)}(f) \right) = \varepsilon$ and hence $L_\phi(\mu, X)$ is not strongly proximinal, which is a contradiction.

3. Conclusion

We conclude that if $Y$ is a separable proximinal subspace of $X$, then $Y$ is strongly proximinal in $X$ if and only if $L_\phi(\mu, Y)$ is strongly proximinal in $L_\phi(\mu, X)$.

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