Annals of Pure and Applied Mathematics
Vol. 24, No. 2, 2021, 83-97
ISSN: 2279-087X (P), 2279-0888(online)
Published on 14 October 2021
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v24n2a01844

Annals of Pure and Applied Mathematics

Three Theorems on the Goldbach Conjecture, and the Intimate Prime-Pairs

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Dedicated to the memory of my late father:
Zhiyun Liu (1930-2020), professor of higher education

To pose good unsolved problems is a difficult art
— Richard K. Guy, *Unsolved Problems in Number Theory*

Received 26 August 2021; accepted 12 October 2021

Abstract. Using a given $\pi_1 < \pi_2 < \cdots < \pi_\mu \cdots$, we define the nominal-prime, what is the relationship to prime? Related to this is homotopy property, and we introduce a number of new algebraic structure, which including great substitutions involving the element entanglements. We'll take it as a pure algebraic application with new ideas to create a complete algebra solution, which is for the first time we have provided proof not only that the Goldbach conjecture of the form $2P = (p + q)$ called a quasi-Goldbach conjecture, but also that if p is a prime, then there is always two primes between p^2 and $(p+1)^2$, congeneric or symmetric with between $(p-1)^2$ and p^2. As a corollary, we obtain that there are infinitely many intimate prime-pairs I_P, I'_p; we also define the number of intimate prime-pairs $\pi(I_p)$, $\pi(I'_p)$. Using right (left) interval theorem, we raises two problems and several conjectures for the sequence of prime numbers.

Keywords: Goldbach conjecture, generalized primes, twin primes, infinite sets

AMS Mathematics Subject Classification (2010): 11A41, 11N80, 11P32

1. Introduction

In 1742, Goldbach suggested roughly hypothesis to Euler (it raises a tricky arithmetical problem) that is provocative ancient yet modern-day conjecture:

Conjecture 1.1. (Goldbach conjecture) *Every even number greater than four is the sum of two primes.*

One of the most celebrated paper — namely the Vinogradov [1] paper is that we know Goldbach's theorem holds for almost all even integers. The prime estimation method for in the number theory (which is looking to acquire the Ω-results and O-results) it is to be workable (see [2, 3, 4, 5, 6, 7]). Erdös [8] asks if there are infinitely many primes p such that every even number $\leq p-3$ can be expressed as the difference between two primes each $\leq p$. On the Bertrand's postulate [9], one could expect dramatic improvements. The gaps between consecutive primes have been of perennial interest; especially the twin primes (see [10, 11, 12, 13, 14]). It is interesting that the relation between Diophantine equation and Goldbach conjecture for a particular case of the equation (see [15, 16]). Because of the plausibility of the Goldbach conjecture, it seems likely that 5 is the only odd untouchable number (see [8, 17, 18]). After analysis it's algebra now.

We can easily verify Goldbach identical equation. So our purpose in this paper just has to prove the case for $2P$. We define the *nominal-prime* (denote π_μ, so named because the nominal-prime that be true of "couple of prime", it is quite so solid number and they have the same characteristic in positive factor), where we take the prime pairs. We stipulate that the π of cut with uncountable infinity of nominal-primes for the element entanglements in great substitutions, then one of the major applications of nominal-primes is for the *quasi-Goldbach conjecture*. On the prime gaps problem, we demonstrate that there is always two primes between p^2 and $(p+1)^2$, and continued in like manner give $(p-1)^2 < q < p < P^2$. As a corollary, we define the *intimate prime-pairs* I_P and the number of intimate prime-pairs $\pi(I_p)$, with $p = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 61, 67$. In closing, this does bring up several interesting problems and several conjectures.

2. Nominal-primes

We want to comprehend the irrational numbers form for the real numbers. Here the π is an important object, processes it, and we see it as prime.

Model (Method of digit positional cuts). For π, cut with it and write π_μ, according to its digit positions, and such that

$$\pi_1 < \pi_2 < \cdots < \pi_\mu \quad \cdots, \tag{1}$$

where $\mu = 1, 2, \cdots$, and $\pi_\mu \in \Pi$, Then rational number $\mathbb{Q}$ is a Π set provided that commutative ring from an ordered field $\mathbb{F}$.

Definition 2.1. (Prime sets) P is said to be prime set if $p \in P$, $p > 1$, it has no proper factor in natural numbers.

Definition 2.2. (Nominal-primes) Let Π be an infinite set and let P be a prime set. Assume that for each $\pi_\mu \in \Pi$, $p \in P$ and $p \geq 3$, the map $f: \Pi \longrightarrow P$, defined by

$$f(\pi_\mu) = p. \tag{2}$$

Then we say that π_μ is a cutting point of f. We call π_μ is a *nominal-prime* of prime.

Starts with

$$3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \cdots.$$

Sequences of form

$$\pi_1, \pi_2, \cdots, \pi_\mu, \cdots.$$

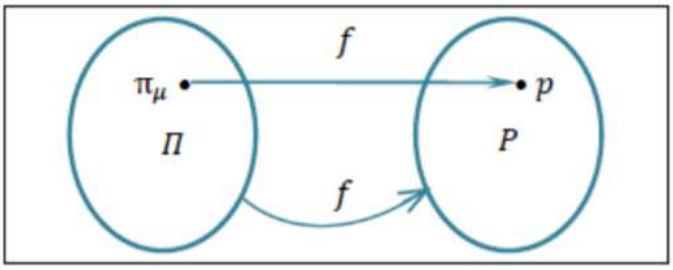

Figure 1: The function of maps Π to P.

Figure 1 represents a function f from Π to P.

Definition 2.3. If P and Π are non-countable sets (infinite sets), we write

$$\nmid P \nmid \; > \aleph_0 \quad \text{and} \quad \nmid \Pi \nmid \; > \aleph_0. \tag{3}$$

Definition 2.4. Let prime set P is equipollent to the nominal-prime set Π if there exists a one-to-one function f with domain P and range Π. we write

$$P \sim \Pi, \tag{4}$$

and we say that f establishes the equipollence P and Π.

Theorem 2.5. (Cardinal numbers theorem) *From Definition* 2.3. *Then*

$$\nmid P \nmid = \nmid \Pi \nmid. \tag{5}$$

Proof: By Definition 2.2 and by Definition 2.4 we have $\nmid P \nmid = 3 = \nmid \Pi \nmid$. □

Definition 2.6. (The Π— P relation) Let Π and P be infinite sets if for each $\pi_\mu \in \Pi$ and $p \in P$. Then

$$\Re = \{(\pi_1, p_1), (\pi_2, p_2), \cdots, (\pi_\mu, p_i), \cdots\}, \tag{6}$$

is a binary relation from Π to P. The ordered pairs in this binary relation are displayed both graphically and in tabular form in tabular form in Figure 2. Let

$$f, g\colon \pi \longrightarrow p.$$

We say that f and g are *homotopy*, and we write $\cong$.

Theorem 2.7. (Homotopy theorem) *Let P be a prime set and let Π be a nominal-prime set. Then*

$$P \cong \Pi. \tag{7}$$

Proof: By Definition 2.2 and by Definition 2.6. For each $p \in P$ and $\pi_\mu \in \Pi$ we have $f(\pi_\mu) = p$. □

Theorem 2.8. (Decimal theorem) *Let $\pi_\mu < \pi$, and let $\hbar$ be a decimal. Then*

$$\sum_{\substack{\mu \in \mathbb{N} \\ \mu=1}}^{n} \pi_\mu = n(\pi - \hbar) \tag{8}$$

Proof: Let $p(n)$ be the proposition that this formula is correct for the integer *n*.

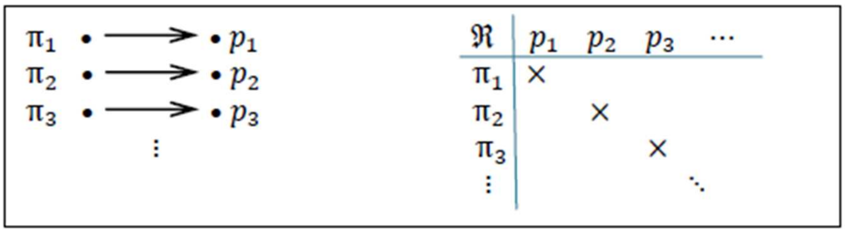

Figure 2: Displaying the ordered pairs (*left panel*) in the relation $\Re$ (*right panel*).

Basis step: $p(1)$ and $p(2)$ are true since

$$3 = 3.1 - 0.1 \quad \text{and} \quad 3 + 3.1 = 2(3.14 - 0.09). \tag{9}$$

Inductive step: We must show that $P(n+1)$ is true when $P(n)$ is assumed to be true. That is, we need to show that

$$\sum_{\substack{\mu\in\mathbb{N}\\ \mu=1}}^{n} \pi_\mu + 1 = (n+1)(\pi - \hbar) \,. \tag{10}$$

This can be done since

$$\begin{aligned} \sum_{\substack{\mu\in\mathbb{N}\\ \mu=1}}^{n} \pi_\mu + 1 &= \sum_{\substack{\mu\in\mathbb{N}\\ \mu=1}}^{n} \pi_\mu + (\pi - \hbar) \\ &= n(\pi - \hbar) + (\pi - \hbar) \\ &= (n+1)(\pi - \hbar). \end{aligned} \tag{11}$$

This finishes the inductive step, which completes the proof. □

Corollary 2.9. *Let* $\pi_{\mu_i} < \pi_{\mu_j} < \pi$, *and let* $\hbar$ *be a decimal. Then*

$$\pi_{\mu_i} + \pi_{\mu_j} = 2(\pi - \hbar). \tag{12}$$

3. Great substitutions

We would prefer to simplify sign, $\pi\ -\pi_\mu$: the meaning of this π, as seen in this Definition 2.2.

Definition 3.1. (Great substitutions) Let S be a substitution group defied on Π and let S' be a substitution group defied on P. Assume that for each $\pi \in \Pi$, define

$$\theta\colon\ \Pi \longrightarrow P \quad \text{and} \quad \psi\colon\ S \longrightarrow S' \tag{13}$$

given by

$$(\pi^{\vartheta})^{\theta} = \left(\pi^{\theta}\right)^{h^{\psi}}. \tag{14}$$

We say that S and S' are isomorphic to substitution (also called *great substitution*), and we write

$$S \doteqdot S', \tag{15}$$

if a substitution isomorphism between them exists. Let $h \in S$ and write

$$h = \binom{\pi_i}{\pi_{j_i}}, \tag{16}$$

the substitution of h is h^{ψ} on ψ . Then

$$h^{\psi} = \begin{pmatrix} \pi_i^{\theta} \\ (\pi_i)^{h^{\psi}} \end{pmatrix} = \begin{pmatrix} \pi_i^{\theta} \\ (\pi_i^{h})^{\theta} \end{pmatrix}. \tag{17}$$

Theorem 3.2. (Length theorem) *Let P be a prime set and let Π be a nominal-prime set. Then*

$$|P| = |N| = |\Pi|. \tag{18}$$

Proof: By Definition 2.2 and by Definition 3.1. For each $p \in P$, $n \in N$ and $\pi \in \Pi$, using the transitive, we are done. □

4. The element entanglement

Definition 4.1. (Element entanglement) Let Ω and Λ be two infinite sets and suppose $\delta\colon \Omega \longrightarrow \Lambda$ is a bijection. We say that δ is an entanglement if there are equivalent at first element, Ω and Λ are *entangled*, and we write

$$\Omega \doteqdot \Lambda, \tag{19}$$

if an entanglement between them exists.

Definition 4.2. Let $a, b, c \in \mathbb{R}$. Then the following properties hold:

$$b = (b \doteqdot a), \tag{20}$$

$$a \doteqdot (b \doteqdot c) = (a \doteqdot b) \doteqdot c, \tag{21}$$

$$a \doteqdot (b + c) = (a \doteqdot b) + (a \doteqdot c). \tag{22}$$

Theorem 4.3. (Entanglement theorem) *Let Π be a nominal-prime set and let P be a prime set. Suppose $f\colon \Pi \longrightarrow P$. Then*

$$\Pi \doteqdot P. \tag{23}$$

Proof: We have $\pi_1 = 3 = p_1$, and by Definition 4.1, as required. We display any tow element entanglement correspondence in Figure 3. □

Theorem 4.4. (Natural entanglement theorem) *Let $N = \{3, 4, 5, \cdots\}$ and let Π be a nominal-prime set. Suppose $\varphi\colon N \longrightarrow \Pi$. Then*

$$N \doteqdot \Pi. \tag{24}$$

Proof: Let $n_1 = 3 = p_1$, by Definition 4.1 and by Theorem 4.3, we have

$$N \doteqdot P \quad \text{and} \quad P \doteqdot \Pi \tag{25}$$

For maps is a transitivity and thus $N \doteqdot \Pi$, as claimed. □

Definition 4.5. (Equivalence of order-set) Let Γ and Ξ be two nonempty ordered sets. If $\Gamma \doteqdot \Xi$. Then we say that

$$\Gamma \sim \Xi. \tag{26}$$

Theorem 4.6. (Natural equivalence theorem) *Let* Π *be a nominal-prime set and let* N

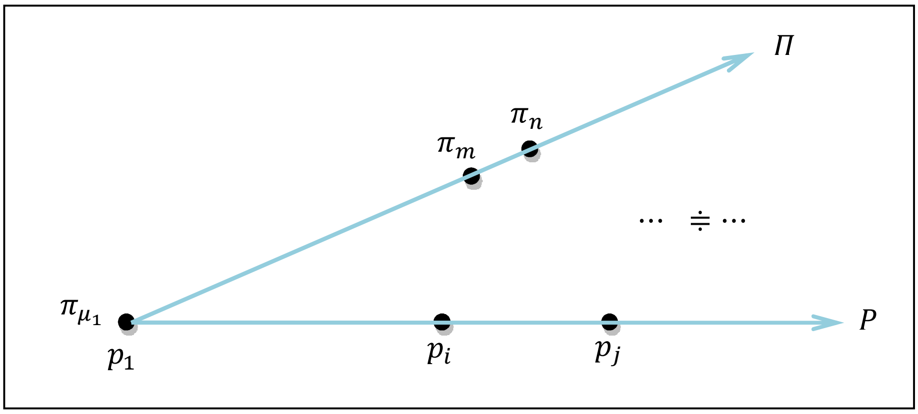

Figure 3: A entangled graph: this figure shows the $\pi_m \doteqdot p_i$ and $\pi_n \doteqdot p_j$.

be a natural number set. If $N \geq 3$. *Then*

$$\Pi \sim N. \tag{27}$$

Proof: By Natural Entanglement Theorem 4.4 and by Definition 2.4, as required. □

Corollary 4.7. *Let* P *be a prime set and let* N *be a natural number set. If* $P,\ N \geq 2$. *Then*

$$P \sim N. \tag{28}$$

Theorem 4.8. *Let,* $a, b \in \mathbb{R}$. *Then the following statements are equivalent:*

$$|a| \doteqdot |b|, \tag{29}$$

$$|a| = |b|, \tag{30}$$

$$a = b. \tag{31}$$

Proof: Since $\doteqdot$ is symmetric and transitive, it is an entanglement relation, and their length is the same. □

5. Further results

Conjecture 5.1. (Quasi-Goldbach conjecture) *The quasi-Goldbach conjecture is whether or not*

$$¿\ 2P = p + q\ ? \tag{32}$$

where p, q and P are primes, and $p < q$ and $P \geq 5$.

Now, the following is an algebraic approach.

Lemma 5.2. (Prime lemma) *Let* p *and* q *be two primes and let* $\hbar \in \mathbb{Z}$. *If* $p_i, \hbar < q_j$. *Then*

$$\sum_{i=1}^{n} p_i = n(q_j - \hbar) \quad \text{for} \ 1 \leq j \leq n. \tag{33}$$

Proof: By the Entanglement Theorem 4.3, we prove like the Decimal Theorem 2.8. □

Corollary 5.3. (Prime corollary) *Let $p < q < r$ in P and suppose $\hbar < r$. Then for $\hbar \in \mathbb{Z}$, we have*

$$2(r - \hbar) < p + q. \tag{34}$$

Proof of Conjecture 5.1. Let r be a prime and suppose $|\hbar| < |r|$ and $\hbar \in \mathbb{Z}$. By the Prime Corollary 5.3 and by the Length Theorem 3.2, we see that

$$N \ngtr r - \hbar \quad \text{and} \quad N \nless r - \hbar. \tag{35}$$

(Note that, using N, which is to gain show) We thus have

$$N = r - \hbar. \tag{36}$$

We need to go back to the Prime Corollary 5.3, hence

$$2N = p + q, \tag{37}$$

then $N = P$ is a prime.

In order to transform it, by the Entanglement Theorem 4.3, we have

$$\pi_m \doteqdot p \quad \text{and} \quad \pi_n \doteqdot q, \tag{38}$$

see Figure 3. By the Natural Entanglement Theorem 4.4, we get

$$p \doteqdot n \quad \text{and} \quad q \doteqdot n, \tag{39}$$

Since

$$|p + q| \doteqdot |n + n|. \tag{40}$$

By Theorem 4.8, in all cases, we can thus write

$$p + q = 2n, \tag{41}$$

since $n \in N$, this yields

$$n = P \subseteq N. \tag{42}$$

We proved for the first time that Conjecture 5.1 is true for double the primes. □

Because of this ingenious proof, we say the same is likely true of other composite numbers to the Goldbach Conjecture 1.1. One of the major applications of actions is for intimate prime-pairs. The key to this is the following theorem.

Theorem 5.4. (Goldbach-Liu theorem) *Let $P \geq 5$ and $p < q$, where P, p and q are primes. We have*

$$2P = p + q. \tag{43}$$

Proof: It's easy to verify Goldbach identity. But technically, we've done the proof, on top. □

Theorem 5.5. (Right interval theorem) *If P, is a prime, then there is always two primes between P^2 and $(P+1)^2$.*

Proof: Let P is prime, we have

$$(P + 1)^2 - P^2 = 2P + 1, \tag{44}$$

by Theorem 5.4, since $2P = p + q$, and hence $P^2 < p < q < (P + 1)^2$. □

As a corollary, for all primes p, we have

$$p^2 < a < b < (p + 1)^2. \tag{45}$$

The number

$$I_p := (a, b), \tag{46}$$

are intimate prime-pairs. Thus there are infinitely many intimate prime-pairs, cf. Appendix A Case 7.1 for more details. We have the following problem and conjecture:

Problem 5.6. *How large is* $\pi(I_p)$ *for arbitrary?*

Conjecture 5.7. *Let* $P < p$ *and* $q < p$, *where* P, p *and* q *are primes. We have*

$$¿\ 2P = p - q\ ? \tag{47}$$

Correspondingly, we give the following theorem.

Theorem 5.8. (Left interval theorem) *If* P *is a prime, then there is always two primes between* $(P-1)^2$ *and* P^2.

Proof: Let P is prime, we have

$$P^2 - (P-1)^2 = 2P - 1. \tag{48}$$

Let $P < p$ and $q < p$, where p and q are primes. By Conjecture 5.7, if $2P = p - q$, then

$$(P-1)^2 < q < p < P^2, \tag{49}$$

and the examples follows. □

As a corollary, for all primes p, we have

$$(p-1)^2 < c < d < p^2, \tag{50}$$

the number

$$I'_p := (c, d), \tag{51}$$

are intimate prime-pairs. Thus there are infinitely many intimate prime-pairs, cf. Appendix A Case 7.2 for more details. For $\pi(I'_p)$, we have the same problem:

Problem 5.9. *How large is* $\pi(I'_p)$ *for arbitrary?*

In intimate prime-pairs I_p and I'_p, ask whether there are twin primes. For example, Table 3 and Table 4 shows twin primes in Appendix A Case 7.3. So we have the following conjecture:

Conjecture 5.10. *Is there always a twin primes between* $(P-1)^2$ *and* $(P+1)^2$*?*

Conjecture 5.11. *Is there always a twin primes between* p *and* $2p$*?*

6. Conclusion

In this paper, we show first that $2P = (p+q)$ is a kind of quasi-Goldbach conjecture, which proof of $2P = (p+q)$ is true. As a result, we see that it is two any prime factors in the range: $P^2 < p < q < (P+1)^2$, $(p-1)^2 < q < p < P^2$. We defines the intimate prime-pairs I_P and I'_p, we also defines the number of intimate prime-pairs $\pi(I_p)$ and $\pi(I'_p)$.

7. Future prospects

We raises two open problems concerning the intimate prime-pairs and several conjectures concerning the twin primes in the sequence of prime numbers, but proof is again hopelessly beyond reach. Such examples be given easily, but they have their own meaning in connection with the Goldbach conjecture and the twin primes conjecture. Initially, one may try to work by computer, continues to add to the list of intimate prime-pairs.

Appendix A: Intimate prime-pairs

Case 7.1. The first intimate prime-pairs I_p are

$$I_2 = (5,\ 7). \quad (52)$$

$$I_3 = (11,\ 13). \quad (53)$$

$$I_5 = (29,\ 31). \quad (54)$$

$$I_7 = (53,\ 59), (53,\ 61), (59,\ 61). \quad (55)$$

$$I_{11} = (127,\ 131), (127,\ 137), (127,\ 139), (131,\ 137), (131,\ 139), (137,\ 139). \quad (56)$$

$$I_{13} = (173, 179), (173, 181), (173, 191), (173, 193), (179, 181), (179, 191), (179,\ 193), (181,\ 191), (181,\ 193), (191,\ 193). \quad (57)$$

$$I_{17} = (293, 307), (293, 311), (293, 313), (293, 317), (307, 311), (307; 313), (307, 317), (311, 313), (311, 317), (313, 317). \quad (58)$$

$$I_{19} = (367, 373), (367, 379), (367, 383), (367, 389), (367, 397), (373, 379), (373, 383), (373, 389), (373, 397), (379, 383), (379, 389), (379, 397), (383, 389), (383, 397), (389, 397). \quad (59)$$

$$I_{23} = (541, 547), (541, 557), (541, 563), (541, 569), (541, 571), (547, 557), (547, 563), (547, 569), (547, 571), (557, 563), (557, 569), (557, 571), (563, 569), (563, 571), (569, 571). \quad (60)$$

$$I_{29} = (853, 857), (853, 859), (853, 863), (853, 877), (853, 881), (853, 883), (853, 887), (857, 859), (857, 863), (857, 877), (857, 881), (857, 883), (857, 887), (859, 863), (859, 877), (859, 881), (859, 883), (859, 887), (863, 877), (863, 881), (863, 883), (863, 887), (877, 881), (877, 883), (877, 887), (881, 883), (881, 887), (883, 887). \quad (61)$$

$I_{31} =$ (967, 971), (967, 977), (967, 983), (967, 991), (967, 997), (967, 1009), (967, 1013), (967, 1019), (967, 1021), (971, 977), (971, 983), (971, 991), (971, 997), (971, 1009), (971, 1013), (971, 1019), (971, 1021), (977, 983), (977, 991), (977, 997), (977, 1009), (977, 1013), (977, 1019), (977, 1021), (983, 991), (983, 997), (983, 1009), (983, 1013), (983, 1019), (983, 1021), (991, 997), (991, 1009), (991, 1013), (991, 1019), (991, 1021), (997, 1009), (997, 1013), (997, 1019), (997, 1021), (1009, 1013), (1009, 1019),

Table 1: Some numbers of intimate prime-pairs $\pi(I_p)$:

p	$\pi(I_p)$
2	1
3	1
5	1
7	3
11	6
13	10
17	10
19	15
23	15
29	28
31	45
37	36
41	55
43	66
47	78
53	66
59	120
61	78
67	105
⋮	⋮

(1009, 1021), (1013, 1019), (1013, 1021), (1019, 1021). (62)

$I_{37} =$ (1373, 1381), (1373, 1399), (1373, 1409), (1373, 1423), (1373, 1427), (1373, 1429), (1373, 1433), (1373, 1439), (1381, 1399), (1381, 1409), (1381, 1423), (1381, 1427), (1381, 1429), (1381, 1433), (1381, 1439), (1399, 1409), (1399, 1423), (1399, 1427), (1399, 1429), (1399, 1433), (1399, 1439), (1409, 1423), (1409, 1427), (1409, 1429), (1409, 1433), (1409, 1439), (1423, 1427), (1423, 1429), (1423, 1433), (1423, 1439), (1427, 1429), (1427, 1433), (1427, 1439), (1429, 1433), (1429, 1439), (1433, 1439). (63)

Let $\pi(I_p)$ be the number of intimate prime-pairs. Table 1 shows $\pi(I_p)$. We can visualize $\pi(I_p)$ more easily with the help of the table, and we find that

$$\pi(I_2) = \pi(I_3) = \pi(I_5) = 1. \quad (64)$$

$$\pi(I_{13}) = \pi(I_{17}) = 10. \quad (65)$$

$$\pi(I_{19}) = \pi(I_{23}) = 15. \tag{66}$$

Table 2: Some numbers of intimate prime-pairs $\pi(I'_p)$:

p	$\pi(I'_p)$
2	1
3	1
5	3
7	6
11	10
13	10
17	21
19	15
23	21
29	36
31	28
37	36
41	66
43	36
47	45
53	120
59	78
61	120
67	105
⋮	⋮

$$\pi(I_{37}) < \pi(I_{31}). \tag{67}$$

$$\pi(I_{43}) = \pi(I_{53}) = 66. \tag{68}$$

$$\pi(I_{53}) < \pi(I_{47}). \tag{69}$$

$$\pi(I_{47}) = \pi(I_{61}) = 78. \tag{70}$$

$$\pi(I_{61}) < \pi(I_{59}). \tag{71}$$

$$\pi(I_{67}) < \pi(I_{59}). \tag{72}$$

Case 7.2. The first intimate prime-pairs I'_p are

$$I'_2 = (2,\ 3). \tag{73}$$

$$I'_3 = (5,\ 7). \tag{74}$$

$I'_5 =$ (17, 19), (17, 23), (19, 23). (75)

$I'_7 =$ (37, 41), (37, 43), (37, 47), (41, 43), (41, 47), (43, 47). (76)

$I'_{11} =$ (101, 103), (101, 107), (101, 109), (101, 113), (103, 107), (103, 109), (103, 113), (107, 109), (107, 113), (109, 113). (77)

$I'_{13} =$ (149, 151), (149, 157), (149, 163), (149, 167), (151, 157), (151, 163), (151, 167), (157, 163), (157, 167), (163, 167). (78)

$I'_{17} =$ (257, 263), (257, 269), (257, 271), (257, 277), (257, 281), (257, 283), (263, 269), (263, 271), (263, 277), (263, 281), (263, 283), (269, 271), (269, 277), (269, 281), (269, 283), (271, 277), (271, 281), (271, 283), (277, 281), (277, 283), (281, 283). (79)

$I'_{19} =$ (331, 337), (331, 347), (331, 349), (331, 353), (331, 359), (337, 347), (337, 349), (337, 353), (357, 359), (347, 349), (347, 353), (347, 359), (349, 353), (349, 359), (353,359). (80)

$I'_{23} =$ (487, 491), (487, 499), (487, 503), (487, 509), (487, 521), (487, 523), (491, 499), (491, 503), (491, 509), (491, 521), (491, 523), (499, 503), (499, 509), (499, 521), (499, 523), (503, 509), (503, 521), (503, 523), (509, 521), (509, 523), (521, 523). (81)

$I'_{29} =$ (787, 797), (787, 809), (787, 811), (787, 821), (787, 823), (787, 827), (787, 829), (787, 839), (797, 809), (797, 811), (797, 821), (797, 823), (797, 827), (797, 829), (797, 839), (809, 811), (809, 821), (809, 823), (809, 827), (809, 829), (809, 839), (811, 821), (811, 823), (811, 827), (811, 829), (811, 839), (821, 823), (821, 827), (821, 829), (821, 839), (823, 827), (823, 829), (823, 839), (827, 829), (827, 839), (829, 839). (82)

$I'_{31} =$ (907, 911), (907, 919), (907, 929), (907, 937), (907, 941), (907, 947), (907, 953), (911, 919), (911, 929), (911, 937), (911, 941), (911, 947), (911, 953), (919, 929), (919, 937), (919, 941), (919, 947), (919, 953), (929, 937), (929, 941), (929, 947), (929, 953), (937, 941), (937, 947), (937, 953), (941, 947), (941, 953), (947, 953). (83)

$I'_{37} =$ (1297, 1301), (1279, 1303), (1279, 1307), (1279, 1319), (1279, 1321), (1279, 1327), (1279, 1361), (1279, 1367), (1301, 1303), (1301, 1307), (1301, 1319), (1301, 1321), (1301, 1327), (1301, 1361), (1301, 1367), (1303, 1307), (1303, 1319), (1303, 1321), (1303, 1327), (1303, 1361), (1303, 1367), (1307, 1319), (1307, 1321), (1307, 1327), (1307, 1361), (1307, 1367), (1319, 1321), (1319, 1327), (1319, 1361), (1319, 1367), (1321, 1327), (1321, 1361), (1321, 1367), (1327, 1361), (1327, 1367), (1361, 1367). (84)

Let $\pi(I_p')$ be the number of intimate prime-pairs. Table 2 shows $\pi(I_p')$. Next we

Table 3: Some twin primes in I_p.

I_p	Twin primes $d_p := 2$
I_2	(5, 7)
I_3	(11, 13)
I_5	(29, 31)
I_7	(59, 61)
I_{11}	(137, 139)
I_{13}	(179, 181), (191, 193)
I_{17}	(311, 313)
I_{19}	–
I_{23}	(569, 571)
I_{29}	(857, 859), (881, 883)
I_{31}	(1019, 1021)
I_{37}	(1427, 1429)
I_{41}	(1697, 1699), (1721, 1723)
I_{43}	(1877, 1879), (1931, 1933)
I_{47}	(2237, 2239), (2267, 2269)
I_{53}	–
I_{59}	(3527, 3529), (3539, 3541), (3557, 3559), (3581, 3583)
I_{61}	(3821, 3823)
I_{67}	(4517, 4519), (4547, 4549)
⋮	⋮

can visualize $\pi(I_p')$ more easily with the help of the table, and we find that

$$\pi(I_2') = \pi(I_3') = 1. \tag{85}$$

$$\pi(I_{11}') = \pi(I_{13}') = 10. \tag{86}$$

$$\pi(I_{17}') = \pi(I_{23}') = 21. \tag{87}$$

$$\pi(I_{19}') < \pi(I_{17}'). \tag{88}$$

$$\pi(I_{31}') < \pi(I_{29}'). \tag{89}$$

$$\pi(I_{37}') = \pi(I_{43}') = 36. \tag{90}$$

$$\pi(I_{43}') < \pi(I_{41}'). \tag{91}$$

$$\pi(I_{47}') < \pi(I_{41}'). \tag{92}$$

$$\pi(I'_{53}) = \pi(I'_{61}) = 120. \tag{93}$$

Table 4: Some twin primes in I'_p.

I'_p	Twin primes $d_p := 2$
I'_3	(5, 7)
I'_5	(17, 19)
I'_7	(41, 43)
I'_{11}	(101, 103), (107, 109)
I'_{13}	(149, 151)
I'_{17}	(269, 271), (281, 283)
I'_{19}	(347, 349)
I'_{23}	(521, 523)
I'_{29}	(809, 811), (821, 823), (827, 829)
I'_{31}	—
I'_{37}	(1301, 1303), (1319, 1321)
I'_{41}	(1607, 1609), (1667, 1669)
I'_{43}	(1787, 1789)
I'_{47}	(2129, 2131), (2141, 2143)
I'_{53}	(2711, 2713), (2729, 2731), (2789, 2791), (2801, 2803)
I'_{59}	(3371, 3373), (3389, 3391), (3461, 3463), (3467, 3469)
I'_{61}	(3671, 3673)
I'_{67}	(4421, 4423), (4481, 4483)
$\vdots$	$\vdots$

$$\pi(I'_{59}) < \pi(I'_{53}). \tag{94}$$

$$\pi(I'_{67}) < \pi(I'_{61}). \tag{95}$$

We also find out that

$$\pi(I_2) = \pi(I'_2) = 1. \tag{96}$$

$$\pi(I_3) = \pi(I'_3) = 1. \tag{97}$$

$$\pi(I_{13}) = \pi(I'_{13}) = 10. \tag{98}$$

$$\pi(I_{19}) = \pi(I'_{19}) = 15. \tag{99}$$

$$\pi(I_{37}) = \pi(I'_{37}) = 36. \tag{100}$$

$$\pi(I_{67}) = \pi(I'_{67}) = 105. \tag{101}$$

Case 7.3. Some twin primes I_p and I'_p in Table 3 and Table 4.

The twin primes $d_p := 2$, for instance.

Clearly, only one of the intimate prime-pairs (5, 7) can be

$$I_2 = I'_3. \tag{102}$$

Indeed, the least intimate prime-pairs is (2, 3).

Acknowledgements. The author would like to thank the reviewers for their careful reading of this article and for valuable comments.

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