Annals of Pure and Applied Mathematics
Vol. 24, No. 2, 2021, 123-133
ISSN: 2279-087X (P), 2279-0888(online)
Published on 30 November 2021
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v24n2a03847

Annals of **Pure and Applied Mathematics**

A New Range of Modified Gamma and Beta Functions

Naresh Dudi

Department of Mathematics and Statistics
Jai Narain Vyas University, Jodhpur
Rajasthan 342001, India
E-mail: nareshdudi@yahoo.co.in

Received 9 October 2021; accepted 28 November 2021

Abstract. The principle aim of this paper is obtain a new range of modified Gamma and Beta function by using the function $E_{K_1,K_2}^{K_3}(-t)=\frac{\sum_{r=0}^{\infty}(-1)^r t^r (k_3)_r}{\Gamma(k_1 r+k_2)\, r!}$, $\left(k_1, k_2, k_3 \in \mathbb{C}, Re(k_1)>0, Re(k_2)>0, Re(k_3)>0\right)$ introduced by the Prabhakar [1] in 1971. Inspired essential by the success of the applications of the generalized Gamma and Beta functions in many areas of science and engineering, the author present, in a unified manner, a detailed account or rather a brief survey of a new rang of modified Gamma and Beta functions, first and second summation relations, various functional, symmetric, Mellin transforms and integral representations are obtained. Furthermore, mean, variance and the moment generating function for the beta distribution of the new range.

Keywords: Gamma function, Beta function, Modified Mittage-Leffler Function, Modified Gamma Function, Modified Beta Function, Beta distributions.

AMS Mathematics Subject Classification (2010): 33B15

1. Introduction

The Classical Euler gamma and beta functions [2] are given by:

$$B(\lambda_1,\lambda_2)=\int_0^{\infty} t^{\lambda_1}(1-t)^{\lambda_2-1}dt=\frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda_1+\lambda_2)} \tag{1}$$

$$\Gamma(\lambda_1)=\int_0^{\infty} t^{\lambda_1-1}e^{-t}dt, Re(\lambda_1)>0, Re(\lambda_2)>0. \tag{2}$$

Classical Euler gamma and beta functions with their connection with Macdonald, error and Whittaker functions [2,3] was extended as follows:

$$\Gamma_{\varpi}(x)=\int_0^{\infty} t^{\lambda_1-1}exp\left(-t-\frac{\varpi}{t}\right)dt. \tag{3}$$

where $Re(\lambda_1)>0, \varpi>0,$
and

$$B\left(\lambda_{1,}\lambda_{2,}\varpi\right) = \int_0^1 t^{\lambda_1-1}\,(1-t)^{\lambda_2-1} exp\left(-\frac{\varpi}{t(1-t)}\right)\,dt. \tag{4}$$

where $Re(\lambda_1) > 0, Re(\lambda_2) > 0, Re(\varpi) > 0$

In 2014, Lee et al [4] generalized beta function given Chaudhry et al [3] as follows:

$$B\left(\lambda_{1,}\lambda_{2,}\varpi;\omega\right) = \int_0^1 t^{\lambda_1-1}\,(1-t)^{\lambda_2-1} exp\left(-\frac{\varpi}{t^{\omega}(1-t)^{\omega}}\right)\,dt \tag{5}$$

where $Re(\lambda_1) > 0, Re(\lambda_2) > 0, Re(\varpi) > 0, Re(\omega) > 0$

In 2014, Choi et al [5] extended the beta function given by Chaudhry et al [3] as follows:

$$B\left(\lambda_{1,}\lambda_{2,}\varpi_1,\varpi_2\right) = \int_0^1 t^{\lambda_1-1}\,(1-t)^{\lambda_2-1} exp\left(-\frac{\varpi_1}{t}-\frac{\varpi_2}{1-t}\right)\,dt \tag{6}$$

where $Re(\lambda_1) > 0, Re(\lambda_2) > 0, Re(\varpi_1) > 0, Re(\varpi_2) > 0$

In 2011, Ozergin et al [6] presented the following generalizations of gamma and beta functions:

$$\Gamma_{\varpi}^{(k_1,k_2)}(\lambda_1) = \int_0^\infty t^{\lambda_1-1}\ {}_1F_1\left(k_1,k_2;-t-\frac{\varpi}{t}\right)dt \tag{7}$$

where $Re(\lambda_1) > 0, Re(\lambda_2) > 0, Re(\varpi) > 0, Re(k_1) > 0, Re(k_2) > 0$

and
$$\mathrm{B}_{\varpi}^{(k_1,k_2)}\left(\lambda_{1,}\lambda_2\right) = \int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1}\ {}_1F_1\left(k_1,k_2;-\frac{\varpi}{t(1-t)}\right)dt \tag{8}$$

where $Re(\lambda_1) > 0, Re(\lambda_2) > 0, Re(\varpi) > 0, Re(k_1) > 0, Re(k_2) > 0$

In 2013, Parmar [7] generalized the result obtained by Ozergin et al in [6] as follows:

$$\Gamma_{\varpi}^{(k_1,k_2;\omega)}(\lambda_1) = \int_0^\infty t^{\lambda_1-1}\ {}_1F_1\left(k_1,k_2;-t^{\omega}-\frac{\varpi}{t^{\omega}}\right)dt \tag{9}$$

where $Re(\lambda_1) > 0, Re(\omega) > 0, Re(\varpi) > 0, Re(k_1) > 0, Re(k_2) > 0$

and
$$\mathrm{B}_{\varpi}^{(k_1,k_2;\omega)}(\lambda_1,\lambda_2) = \int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1}\ {}_1F_1\left(k_1,k_2;-\frac{\varpi}{t^{\omega}(1-t)^{\omega}}\right)dt \tag{10}$$

where $Re(\lambda_1) > 0, Re(\lambda_2) > 0, Re(\varpi) > 0, Re(k_1) > 0, Re(k_2) > 0, Re(\omega) > 0$

In 2015, Agarwal et al [8] used beta function introduced by Parmar [7]to develop two and three variables hypergeometric functions as follows:

$$F_{1,\varpi}^{(k_1,k_2;\omega)}\left(\lambda_{1,}\lambda_{2,}\lambda_3;\lambda_4;x,y\right) = \sum_{r,s=0}^{\infty}(\lambda_2)_r(\lambda_3)_s\,\frac{\mathrm{B}_{\varpi}^{(k_1,k_2;\omega)}(\lambda_1+r+s,\lambda_4-\lambda_1)}{B\left(\lambda_{1,}\lambda_4-\lambda_1\right)}\frac{x^r}{r!}\frac{y^s}{s!} \tag{11}$$

$$(max\{|x|,|y|\} < 1; Re(\varpi) \geq 0; min\{Re(k_1) \geq 0, Re(k_2) \geq 0, Re(\omega) \geq 0\})$$

$$F_{2,\varpi}^{(k_1,k_2,k_1^1,k_2^1;\omega)}\left(\lambda_{1,}\lambda_{2,}\lambda_3;\lambda_4;\lambda_5;x,y\right) = \sum_{r,s=0}^{\infty}(\lambda_1)_{r+s}\,\frac{\mathrm{B}_{\varpi}^{(k_1,k_2;\omega)}(\lambda_2+r,\lambda_4-\lambda_2)}{B\left(\lambda_{2,}\lambda_4-\lambda_2\right)}\frac{x^r}{r!}\frac{y^s}{s!}$$

$$\times\frac{\mathrm{B}_{\varpi}^{\left(k_1^1,k_2^1;\omega\right)}(\lambda_3+s,\lambda_5-\lambda_3)}{B\left(\lambda_{3,}\lambda_5-\lambda_3\right)}\frac{x^r}{r!}\frac{y^s}{s!} \tag{12}$$

$$(max\{|x|,|y|\} < 1; Re(\varpi) \geq 0; min\{Re(k_1) \geq 0, Re(k_2) \geq 0, Re(k_1^1) \geq 0, Re(k_2^1) \geq 0, Re \geq 0\})$$

$$F_{D,\varpi}^{(3;k_1,k_2;\omega)}\left(\lambda_{1,}\lambda_{2,}\lambda_3;\lambda_4,\lambda_5;x,y,z\right) = \sum_{r,s,t=0}^{\infty}(\lambda_2)_r(\lambda_3)_s(\lambda_4)_t\,\frac{\mathrm{B}_{\varpi}^{(k_1,k_2;\omega)}(\lambda_1+r+s+t,\lambda_5-\lambda_1)}{B\left(\lambda_{1,}\lambda_5-\lambda_1\right)}\frac{x^r}{r!}\frac{y^s}{s!}\frac{z^t}{t!} \tag{13}$$

$$(max\{|x|,|y|,|z|\} < 1; Re(\varpi) \geq 0; min\{Re(k_1) \geq 0, Re(k_2) \geq 0, Re(\omega) \geq 0\})$$

In 2017, Pucheta [9] introduced an extended gamma and functions using one parameter MIttage-Leffler function below:

$$\Gamma^{k_1}(\lambda_1) = \int_0^1 t^{\lambda_1 - 1} E_{k_1}(-t) dt \quad (14)$$

and

$$B_{\varpi}^{k_1}(\lambda_1, \lambda_2) = \int_0^1 t^{\lambda_1 - 1} (1-t)^{\lambda_2 - 1} E_{k_1}\big(-\varpi\, t(1-t)\big) dt \quad (15)$$

Recently, many generalizations, modifications, extensions and variants of gamma and beta functions [10-29] have been proposed.

In this paper we generalized the result obtained by Pucheta [9] given in equations (14) and (15) by using the function introduced by Prabhakar [1]

$$E_{k_1,k_2}^{k_3}(-t) = \sum_{r=0}^{\infty} \frac{(-1)^r (k_3)_r}{\Gamma(k_1 r + k_2)} \frac{t^r}{r!},$$

$$(k_1, k_2, k_3 \in \mathbb{C}\,; Re(k_1) > 0, Re(k_2) > 0, Re(k_3) > 0\,), \quad (16)$$

where $(k_3)_r$ is the Pochhammer symbol (Rainville [37])

$$(k_3)_0 = 1, \quad (k_3)_r = k(k+1)(k+2)\ldots(k+r-1.)$$

The paper is organised as follows: Section one comprises introduction and related literature. Section two covers Mellin transform functional, symmetry and summation relations. Section third discusses integral representations. Section four contains some statistical applications.

2. Main result

In this part, we introduced a new range of the modified gamma and beta function with their properties such as functional relations and Mellin transform.

Definition 2.1. Let $k_1, k_2, k_3 \in \mathbb{R}^+, \lambda_1 \in \mathbb{C}$ such that $Re(\lambda_1) > 0$.Then, the extended gamma function is given by :

$$\Gamma_{k_3}^{k_1,k_2}(\lambda_1) = \int_0^{\infty} t^{\lambda_1 - 1} E_{k_1,k_2}^{k_3}(-t)\, dt \quad (17)$$

where $E_{k_1,k_2}^{k_3}(-t)$ is the function introduced by Prabhakar [1] in the form of

$$E_{k_1,k_2}^{k_3}(-t) = \sum_{r=0}^{\infty} \frac{(-1)^r (k_3)_r}{\Gamma(k_1 r + k_2)} \frac{t^r}{r!} \quad (18)$$

Remark 2.1

1. If $k_2 = k_3 = 1$, then $\Gamma_{k_3}^{k_1,k_2}(\lambda_1) = \Gamma^{k_1}(\lambda_1)$ given in equation (14).
2. If $k_1 = k_2 = k_3 = 1$, then $\Gamma_{k_3}^{k_1,k_2}(\lambda_1) = \Gamma(\lambda_1)$ given in equation (2).

Lemma 2.1. Let $k_1, k_2, k_3 \in \mathbb{R}^+, \lambda_1 \in \mathbb{C}$,

$$\Gamma_{k_3}^{k_1,k_2}(\lambda_1) = \frac{\Gamma(\lambda_1 + 1)\,\Gamma\big(1 - (\lambda_1 + 1)\big)}{\Gamma\big(k_3 - k_1(1 + \lambda_1)\big)\,\Gamma\big(k_2 - k_1(1 + \lambda_1)\big)} \quad (19)$$

Proof: Let $\vartheta = \lambda_1 + 1$

$$\Gamma_{k_3}^{k_1,k_2}(\lambda_1 + 1) = \Gamma_{k_3}^{k_1,k_2}(\vartheta) = \int_0^{\infty} t^{\vartheta - 1} E_{k_1,k_2}^{k_3}(-t)\, dt$$

$$=M\left\{E^{k_3}_{k_1,k_2}(-t)\right\}(\vartheta)=\frac{\Gamma(\vartheta)\,\Gamma(1-\vartheta)}{\Gamma(k_2-k_1\vartheta)\,\Gamma(k_3-k_1\vartheta)} \tag{20}$$

where $M\left\{E^{k_3}_{k_1,k_2}(-t)\right\}(\vartheta)$ is Mellin transforms. Replacing $\vartheta=\lambda_1+1$ we obtain the required result.

Definition 2.2. Let $\varpi>0, k_1,k_2,k_3\in\mathbb{R}^+$ *and* $\lambda_1,\lambda_2\in\mathbb{C}$ *such that* $Re(\lambda_1)>0, Re(\lambda_2)>0$. Then the new extended beta function is given by

$$B^{k_1,k_2}_{\varpi,k_3}(\lambda_1,\lambda_2)=\int_0^1 t^{\lambda_1-1}\,(1-t)^{\lambda_2-1}E^{k_3}_{k_1,k_2}\big(-\varpi\, t(1-t)\big)dt \tag{21}$$

Remarks 2.2.

1. If $k_2=k_3=1$, then $B^{k_1,k_2}_{\varpi,k_3}(\lambda_1,\lambda_2)=B^{k_1}_{\varpi}(\lambda_1,\lambda_2)$ given in equation (15).
2. If $k_1=k_2=k_3=1$, and $\varpi=0$, then $B^{k_1,k_2}_{\varpi,k_3}(\lambda_1,\lambda_2)=B(\lambda_1,\lambda_2)$ given in equation (1).

Theorem 2.2. (Functional Relation) Let $\varpi\geq 0, k_1,k_2,k_3\in\mathbb{R}^+$ and $\lambda_1,\lambda_2\in\mathbb{C}$ such that $Re(\lambda_1+1)>0, Re(\lambda_2+1)>0$. Then, the new extended functional relation is given by:

$$B^{k_1,k_2}_{\varpi,k_3}(\lambda_1,\lambda_2+1)+B^{k_1,k_2}_{\varpi,k_3}(\lambda_1+1,\lambda_2)=B^{k_1,k_2}_{\varpi,k_3}(\lambda_1,\lambda_2) \tag{22}$$

Proof:

$$B^{k_1,k_2}_{\varpi,k_3}(\lambda_1,\lambda_2+1)+B^{k_1,k_2}_{\varpi,k_3}(\lambda_1+1,\lambda_2)$$
$$=\int_0^1 t^{\lambda_1-1}\,(1-t)^{\lambda_2}E^{k_3}_{k_1,k_2}\big(-\varpi\, t(1-t)\big)dt$$
$$+\int_0^1 t^{\lambda_1}\,(1-t)^{\lambda_2-1}E^{k_3}_{k_1,k_2}\big(-\varpi\, t(1-t)\big)dt$$

$$B^{k_1,k_2}_{\varpi,k_3}(\lambda_1,\lambda_2+1)+B^{k_1,k_2}_{\varpi,k_3}(\lambda_1+1,\lambda_2)$$
$$=\int_0^1 [t^{-1}+(1-t)^{-1}]\,t^{\lambda_1}(1-t)^{\lambda_2}E^{k_3}_{k_1,k_2}\big(-\varpi\, t(1-t)\big)dt$$

$$B^{k_1,k_2}_{\varpi,k_3}(\lambda_1,\lambda_2+1)+B^{k_1,k_2}_{\varpi,k_3}(\lambda_1+1,\lambda_2)=\int_0^1 t^{\lambda_1-1}\,(1-t)^{\lambda_2-1}E^{k_3}_{k_1,k_2}\big(-\varpi\, t(1-t)\big)dt$$

$$B^{k_1,k_2}_{\varpi,k_3}(\lambda_1,\lambda_2+1)+B^{k_1,k_2}_{\varpi,k_3}(\lambda_1+1,\lambda_2)=B^{k_1,k_2}_{\varpi,k_3}(\lambda_1,\lambda_2)$$

Theorem 2.3. (Symmetry Relation) Let $\varpi\geq 0, Re(\lambda_1)>0$ *and* $Re(\lambda_2)>0$. Then, the new extended beta symmetry relation is given by:

$$B^{k_1,k_2}_{\varpi,k_3}(\lambda_1,\lambda_2)=B^{k_1,k_2}_{\varpi,k_3}(\lambda_2,\lambda_1) \tag{23}$$

Proof: On substituting $t=1-u$ in equation (21) and interchanging the variables, we obtain the required result.

Theorem 2.4. (Mellin Transform). Let $\varpi \geq 0,, k_1, k_2, k_3 \in \mathbb{R}^+, \vartheta \in \mathbb{C}$ such that $Re(\vartheta) > 0, Re(\lambda_1 - \vartheta) > 0$ and $Re(\lambda_2 - \vartheta) > 0$. Then , the new extended Mellin transform is given by

$$M\left[B_{\varpi,k_3}^{k_1,k_2}(\lambda_1, \lambda_2): \vartheta\right] = B(\lambda_1 - \vartheta, \lambda_2 - \vartheta)\, \Gamma_{k_3}^{k_1,k_2}(\lambda_1) \tag{24}$$

Proof:

$$M\left[B_{\varpi,k_3}^{k_1,k_2}(\lambda_1, \lambda_2): \vartheta\right] = \int_0^\infty \varpi^{\vartheta-1}\left(\int_0^1 t^{\lambda_1-1}\,(1-t)^{\lambda_2-1} E_{k_1,k_2}^{k_3}\big(-\varpi\, t(1-t)\big)dt\right) d\varpi \tag{25}$$

Using uniform convergence of integral we can interchange the order of integration equation (25) yield

$$M\left[B_{\varpi,k_3}^{k_1,k_2}(\lambda_1, \lambda_2): \vartheta\right] = \int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1} \int_0^\infty \varpi^{\vartheta-1}\, E_{k_1,k_2}^{k_3}\big(-\varpi\, t(1-t)\big)dt d\varpi \tag{26}$$

Substituting $\varpi = u\, t^{-1}(1-t)^{-1}$ and $t = w$ then $d\varpi = t^{-1}(1-t)^{-1}du$ and $\mathrm{dt} = dw$, equation (26) gives

$$\mathrm{M}\left[B_{\varpi,k_3}^{k_1,k_2}(\lambda_1, \lambda_2): \vartheta\right] = \int_0^1 w^{\lambda_1-\vartheta-1}(1-w)^{\lambda_2-\vartheta-1}dw \int_0^\infty u^{\vartheta-1}\, E_{k_1,k_2}^{k_3}(-u)du$$

$$M\left[B_{\varpi,k_3}^{k_1,k_2}(\lambda_1, \lambda_2): \vartheta\right] = \int_0^1 w^{\lambda_1-\vartheta-1}(1-w)^{\lambda_2-\vartheta-1}dw\ \Gamma_{k_3}^{k_1,k_2}(\vartheta)$$

$$M\left[B_{\varpi,k_3}^{k_1,k_2}(\lambda_1, \lambda_2): \vartheta\right] = B(\lambda_1 - \vartheta, \lambda_2 - \vartheta)\, \Gamma_{k_3}^{k_1,k_2}(\vartheta) \tag{27}$$

Remarks 2.3.

1. Putting $k_2 = k_3 = 1$, in equation (24) , we get
   $M\left[B_{\varpi,k_3}^{k_1,k_2}(\lambda_1, \lambda_2): \vartheta\right] = B(\lambda_1 - \vartheta, \lambda_2 - \vartheta)\Gamma^{k_1}(\vartheta)$ given in Pucheta [9].
2. Putting $k_1 = k_2 = k_3 = 1$, in equation (24) , we get
   $\int_0^\infty B_{\varpi,k_3}^{k_1,k_2}(\lambda_1, \lambda_2) = B(\lambda_1 - 1, \lambda_2 - 1)$ given in Chaudhury [2].

3. Integral representations

Theorem 3.1. The following integral transforms holds true:

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1, \lambda_2) = 2\int_0^{\frac{\pi}{2}} cos^{2\lambda_1-1}\phi\, sin^{2\lambda_2-1}\phi\, E_{k_1,k_2}^{k_3}(-\varpi\, cos^2\phi\, sin^2\phi)d\phi \tag{28}$$

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1, \lambda_2) = n\int_0^1 t^{n\,\lambda_1-1}\,(1-t^n)^{\lambda_2-1} E_{k_1,k_2}^{k_3}\big(-\varpi\, t^n(1-t^n)\big)dt \tag{29}$$

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1, \lambda_2) = \frac{1}{\alpha^{\lambda_1+\lambda_2-1}}\int_0^\alpha t^{\lambda_1-1}\,(\alpha-t)^{\lambda_2-1} E_{k_1,k_2}^{k_3}\left(\frac{-\varpi\, t(\alpha-t)}{\alpha^2}\right)dt \tag{30}$$

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1, \lambda_2) = (1+\alpha)^{\lambda_1-1}\alpha^{\lambda_2-1}\int_0^1 \frac{t^{\lambda_1-1}(1-t)^{\lambda_2-1}}{(t+\alpha)^{\lambda_1+\lambda_2}} E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,\alpha(1+\alpha)t(1-t)}{(t+\alpha)^2}\right)dt \tag{31}$$

Proof: In equation (21), putting $t = cos^2\phi$ then $dt = -2\, cos\phi\, sin\phi\, d\phi$, when $t = 0: \phi = \frac{\pi}{2}$ and $t = 1: \phi = 0$. Therefore

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1, \lambda_2) = \int_{\frac{\pi}{2}}^0 cos^{2\lambda_1-2}\phi\, sin^{2\lambda_2-2}\phi\, E_{k_1,k_2}^{k_3}(-\varpi\, cos^2\phi\, sin^2\phi)(-2\, cos\phi\, sin\phi\, d\phi) \tag{32}$$

On simplifying we get required result.

In equation (21), putting $t = u^n$ then $dt = n\,u^{n-1}du$ when $t = 0{:}\,u = 0$ and $t = 1{:}\,u = 1$. Therefore

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \int_0^1 u^{n(\lambda_1-1)}\,(1-u^n)^{\lambda_2-1}E_{k_1,k_2}^{k_3}\big(-\varpi\,u^n(1-u^n)\big)nu^{n-1}du \tag{33}$$

On simplifying, we obtained the required result.

In equation (21), putting $t = \frac{u}{\alpha}$ then $dt = \frac{du}{\alpha}$ when $t = 0{:}\,u = 0$ and $t = 1{:}\,u = \alpha$. Therefore

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \int_0^\alpha \left(\frac{u}{\alpha}\right)^{\lambda_1-1}\left(\frac{\alpha-u}{\alpha}\right)^{\lambda_2-1}E_{k_1,k_2}^{k_3}\left(-\varpi\frac{u}{\alpha}\left(\frac{\alpha-u}{\alpha}\right)\right)\frac{du}{\alpha} \tag{34}$$

On simplifying, we obtained the required result.

In equation (21), putting $t = \frac{(1+\alpha)u}{(u+\alpha)}$ then $dt = \frac{\alpha(1+\alpha)}{(u+\alpha)^2}du$ when $t = 0{:}\,u = 0$ and $t = 1{:}\,u = 1$. Therefore

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \int_0^1 \left(\frac{(1+\alpha)\,u}{u+\alpha}\right)^{\lambda_1-1}\left(\frac{\alpha(1-u)}{u+\alpha}\right)^{\lambda_2-1}E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,\alpha(1+\alpha)u(1-u)}{(u+\alpha)^2}\right)\times\frac{\alpha(1+\alpha)}{(u+\alpha)^2}\,du \tag{35}$$

On simplifying, we get the required result.

Theorem 3.2. The following integral transformations holds true:

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \int_0^\infty \frac{t^{\lambda_1-1}}{(1+t)^{\lambda_1+\lambda_2}}E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,t}{(1+t)^2}\right)dt \tag{36}$$

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \frac{1}{2}\int_0^\infty \frac{t^{\lambda_1-1}+t^{\lambda_2-1}}{(1+t)^{\lambda_1+\lambda_2}}E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,t}{(1+t)^2}\right)dt \tag{37}$$

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \int_0^1 \frac{t^{\lambda_2-1}}{(1+t)^{\lambda_1+\lambda_2}}E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,t}{(1+t)^2}\right)dt \tag{38}$$

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \alpha^{\lambda_1}\beta^{\lambda_2}\int_0^\infty \frac{t^{\lambda_1-1}}{(\beta+\alpha t)^{\lambda_1+\lambda_2}}E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,\alpha\,\beta\,t}{(\beta+\alpha t)^2}\right)dt \tag{39}$$

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = 2\alpha^{\lambda_1}\beta^{\lambda_2}\int_0^{\frac{\pi}{2}} \frac{sin^{2\lambda_1-1}\phi\;cos^{2\lambda_2-1}\phi}{(cos\phi+asin^2\phi)^{\lambda_1+\lambda_2}}E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,\alpha\,\beta tan^2\phi}{(\beta+\alpha tan^2\phi)^2}\right)d\phi \tag{40}$$

Proof: In equation (21), putting $t = \frac{u}{(1+u)}$ then $dt = \frac{du}{(1+u)^2}$, when $t = 0{:}\,u = 0$ and $t = 1{:}\,u \to \infty$. Therefore

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \int_0^\infty \frac{u^{\lambda_1-1}}{(1+u)^{\lambda_1-1}}\frac{1}{(1+u)^{\lambda_2-1}}E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,u}{(1+u)^2}\right)\frac{du}{(1+u)^2}$$

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \int_0^\infty \frac{u^{\lambda_1-1}}{(1+u)^{\lambda_1+\lambda_2}}E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,u}{(1+u)^2}\right)du \tag{41}$$

On interchanging the variables, we obtained the required result. Using symmetry property in (41) we obtain:

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \int_0^\infty \frac{u^{\lambda_2-1}}{(1+u)^{\lambda_1+\lambda_2}}E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,u}{(1+u)^2}\right)du \tag{42}$$

On adding (41) and (42) we obtain the required result. Using equation (41) we get

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \int_0^1 \frac{u^{\lambda_1-1}}{(1+u)^{\lambda_1+\lambda_2}}E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,u}{(1+u)^2}\right)du + \int_1^\infty \frac{u^{\lambda_1-1}}{(1+u)^{\lambda_1+\lambda_2}}E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,u}{(1+u)^2}\right)du \tag{43}$$

Setting $u = t^{-1}$ then $du = -t^{-2}dt$ when $u = 1: t = 1$ and $u \to \infty: t = 0$. On the second integral of right hand side of equation (43) gives the desired result.

In equation (36), using $t = \frac{\alpha}{\beta}u$ then $dt = \frac{\alpha}{\beta}\, du$ when $x = 0: u = 0$ and $t \to \infty: u \to \infty$. Therefore,

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \int_0^\infty \frac{\left(\frac{\alpha}{\beta}u\right)^{\lambda_1-1}}{\left(1+\frac{\alpha}{\beta}u\right)^{\lambda_1+\lambda_2}} E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\left(\frac{\alpha}{\beta}u\right)}{\left(1+\frac{\alpha}{\beta}u\right)^2}\right)\frac{\alpha}{\beta}du \quad (44)$$

On simplifying, we obtain the desired result.

In equation (39), putting $t = tan^2\phi$ then $dx = 2\tan\phi\, sec^2\phi\, d\emptyset$ when $t = 0: \phi = 0$ and $dt \to \infty: \phi \to \frac{\pi}{2}$. Therefore

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \alpha^{\lambda_1}\beta^{\lambda_2}\int_0^{\frac{\pi}{2}} \frac{(tan^2\phi)^{\lambda_1-1}}{(1+\alpha\, tan^2\phi)^{\lambda_1+\lambda_2}} E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,\alpha\,\beta\, tan^2\phi}{(\beta+\alpha tan^2\phi)^2}\right) 2\, tan\phi\, sec\phi d\phi \quad (45)$$

On simplifying, we get the desired result.

Theorem 3.3. The following integral representation holds true

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \alpha^{\lambda_1}\beta^{\lambda_2}\int_0^1 \frac{t^{\lambda_1-1}(1-t)^{\lambda_2-1}}{(\alpha+(\beta-\alpha)\,t)^{\lambda_1+\lambda_2}} E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,\alpha\,\beta\, t(1-t)}{(\beta+(\alpha-\beta)t)^2}\right)dt \quad (46)$$

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = (\beta+\gamma)^{\lambda_1}\beta^{\lambda_2}\int_0^1 \frac{t^{\lambda_1-1}(1-t)^{\lambda_2-1}}{(\beta+\gamma\, t)^{\lambda_1+\lambda_2}} E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,\alpha\,\beta\, t(1-t)}{(\beta-\gamma t)^2}\right)dt \quad (47)$$

Proof: In equation (21) , putting $\frac{\alpha}{u} - \frac{\beta}{t} = \alpha - \beta$ then $dt = \frac{\alpha\,\beta}{(\alpha+(\beta-\alpha)u)^{\lambda_1+\lambda_2}}du$ when t=0:u=0 and t=1:u=1 give the desired result. Lastly, interchanging α and β in equation (46) and substitute $\alpha - \beta = \gamma$ give equation (47).

Theorem 3.4. The following integral representations holds true:

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = (\beta-\alpha)^{1-\lambda_1-\lambda_2}\int_\alpha^\beta (t-\alpha)^{\lambda_1-1}(\beta-t)^{\lambda_2-1} E_{k_1,k_2}^{k_3}\left(-\frac{\varpi(t-\alpha)\ \ (\beta-t)}{(\beta-\alpha)^2}\right)dt \quad (48)$$

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = 2^{1-\lambda_1-\lambda_2}\int_{-1}^1 (t+1)^{\lambda_1-1}(1-t)^{\lambda_2-1} E_{k_1,k_2}^{k_3}\left(-\frac{\varpi(t+1)\ \ (1-t)}{4}\right)dt \quad (49)$$

Proof: Firstly, in (21) putting $t = \frac{u-\alpha}{\beta-\alpha}$, then $dt = \frac{du}{\beta-\alpha}$, when $t = 0: u = \alpha$ and $t = 1: u = \beta$. Therefore

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \int_\alpha^\beta \frac{(u-\alpha)^{\lambda_1-1}(\beta-u)^{\lambda_2-1}}{(\beta-\alpha)^{\lambda_1-1}(\beta-\alpha)^{\lambda_2-1}} E_{k_1,k_2}^{k_3}\left(-\frac{\varpi(u-\alpha)\ \ (\beta-u)}{(\beta-\alpha)^2}\right)\frac{du}{(\beta-\alpha)}$$

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = (\beta-\alpha)^{1-\lambda_1-\lambda_2}$$

$$\times \int_\alpha^\beta (u-\alpha)^{\lambda_1-1}(\beta-u)^{\lambda_2-1} E_{k_1,k_2}^{k_3}\left(-\frac{\varpi(u-\alpha)\ \ (\beta-u)}{(\beta-\alpha)^2}\right)du \quad (50)$$

On interchanging the variables, we get desired result. Lastly, in (50) putting $\alpha = -1\ and\ \beta = 1,$ we obtained the required result.

Theorem 3.5. The following formula hold.

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \alpha^{\lambda_2}\beta^{\lambda_1}\int_0^1 \frac{t^{\lambda_1-1}(1-t)^{\lambda_2-1}}{(\alpha+(\beta-\alpha)\,t)^{\lambda_1+\lambda_2}} E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,\alpha\,\beta\, t(1-t)}{(\alpha+(\beta-\alpha)t)^2}\right)dt \quad (51)$$

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = (\beta+\gamma)^{\lambda_1}\beta^{\lambda_2}\int_0^1 \frac{t^{\lambda_1-1}(1-t)^{\lambda_2-1}}{(\beta+\gamma\, t)^{\lambda_1+\lambda_2}} E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,\alpha\,\beta\, t(1-t)}{(\beta+\gamma\, t)^2}\right) dt \tag{52}$$

Proof: Firstly in equation (21), putting $\frac{\alpha}{u} - \frac{\beta}{t} = \alpha - \beta$ then $dt = \frac{\alpha\beta}{(\alpha+(\beta-\alpha)u)}\, du$, when $t = 0: u = 0$ and $t = 1: u = 1$, therefore

$$B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2) = \alpha^{\lambda_2-1}\beta^{\lambda_1-1}\int_0^1 \frac{u^{\lambda_1-1}(1-u)^{\lambda_2-1}}{(\alpha+(\beta-\alpha)\,u)^{\lambda_1+\lambda_2}} E_{k_1,k_2}^{k_3}\left(-\frac{\varpi\,\alpha\,\beta\, u(1-u)}{(\alpha+(\beta-\alpha)u)^2}\right) \times \frac{\alpha\,\beta}{(\alpha+(\beta-\alpha)u)^2}\, du \tag{53}$$

On simplifying, we obtain the desired result in equation (53). Lastly, in equation (53) interchanging α and β and later replacing $\alpha - \beta = \gamma$ give the desired result.

Theorem 3.6. The following integral representation formula hold true:

$$\Gamma_{k_3}^{k_1,k_2}(\lambda_1)\Gamma_{k_3}^{k_1,k_2}(\lambda_2) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} r^{2(\lambda_1+\lambda_2-1)} r\cos^{2\lambda_1-1}\theta \sin^{2\lambda_2-1}\theta\, E_{k_1,k_2}^{k_3}(-\varpi\cos^2\theta) E_{k_1,k_2}^{k_3}(-\varpi\sin^2\theta)\, dr\, d\theta \tag{54}$$

Proof: In equation (17), putting $\lambda_1 = n^2$ and $\lambda_2 = m^2$ yield:

$$\Gamma_{k_3}^{k_1,k_2}(\lambda_1) = 2\int_0^{\infty} n^{2\lambda_1-1} t^{\lambda_1-1} E_{k_1,k_2}^{k_3}(-\varpi n^2)\, dn \tag{55}$$

$$\Gamma_{k_3}^{k_1,k_2}(\lambda_2) = 2\int_0^{\infty} m^{2\lambda_2-1} t^{\lambda_1-1} E_{k_1,k_2}^{k_3}(-\varpi m^2)\, dm \tag{56}$$

Multiplying (55) and (56), yields:

$$\Gamma_{k_3}^{k_1,k_2}(\lambda_1)\Gamma_{k_3}^{k_1,k_2}(\lambda_2) = 4\int_0^{\infty}\int_0^{\infty} n^{2\lambda_1-1} m^{2\lambda_2-1} E_{k_1,k_2}^{k_3}(-\varpi n^2) E_{k_1,k_2}^{k_3}(-\varpi m^2)\, dn\, dm \tag{57}$$

Putting $n = r\cos\theta$ and $m = r\sin\theta$ give the result.

Remarks 3.1. If $k_1 = k_2 = k_3 = 1$ and $x = r^2$, then

$\Gamma_{k_3}^{k_1,k_2}(\lambda_1)\Gamma_{k_3}^{k_1,k_2}(\lambda_2) = \Gamma(\lambda_1)\Gamma(\lambda_2) = B(\lambda_1,\lambda_2)\Gamma(\lambda_1+\lambda_2)$ given in [2].

Other integral formulas for related generalized gamma and beta functions are given by Abubakar and Kabara in [35,36].

4. The Beta distribution of $B_{\varpi,k_3}^{k_1,k_2}(\lambda_1,\lambda_2)$

The extended modified beta distribution $B_{\varpi,k_3}^{k_1,k_2}(h,g)$, where h and g satisfy the condition $-\infty < g < \infty$ and $\sigma > 0$ as

$$f(x) = \begin{cases} \frac{1}{B_{\varpi,k_3}^{k_1,k_2}(h,g)} t^{h-1}(1-t)^{g-1} E_{k_1,k_2}^{k_3}\left(-\varpi\, t(1-t)\right), & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases} \tag{58}$$

where $(h, g \in \mathbb{R}, \varpi, k_1, k_2, k_3 \in \mathbb{R}^+)$

The r^{th} moment of X

For any real number, r is given by:

$$E(X^r) = \frac{B^{k_1,k_2}_{\varpi,k_3}(h+r,g)}{B^{k_1,k_2}_{\varpi,k_3}(h,g)}, \ (h, g \in \mathbb{R}, \varpi, k_1, k_2, k_3 \in \mathbb{R}^+) \tag{59}$$

For $r = ,1$ the mean is given by:

$$\mu = E(X) = \frac{B^{k_1,k_2}_{\varpi,k_3}(h+1,g)}{B^{k_1,k_2}_{\varpi,k_3}(h,g)}, (h, g \in \mathbb{R}, \varpi, k_1, k_2, k_3 \in \mathbb{R}^+) \tag{60}$$

The variance of the distribution is given by:

$$\delta^2 = E(X^2) - \{E(X)\}^2$$

$$\delta^2 = \frac{B^{k_1,k_2}_{\varpi,k_3}(h,g)B^{k_1,k_2}_{\varpi,k_3}(h+2,g) - \left\{B^{k_1,k_2}_{\varpi,k_3}(h+1,g)\right\}^2}{\left\{B^{k_1,k_2}_{\varpi,k_3}(h,g)\right\}^2} \tag{61}$$

The moment generating function of the distribution is given by:

$$M(t) = \frac{1}{B^{k_1,k_2}_{\varpi,k_3}(h,g)} \sum_{n=0}^{\infty} B^{k_1,k_2}_{\varpi,k_3}(h+n,g) \frac{t^n}{r!} \tag{62}$$

The cumulative distribution is defined as

$$F(t) = \frac{B^{k_1,k_2}_{\varpi,k_3,t}(h,g)}{B^{k_1,k_2}_{\varpi,k_3}(h,g)} \tag{63}$$

where

$B^{k_1,k_2}_{\varpi,k_3,t}(h,g) = \int_0^1 t^{h-1}(1-t)^{g-1} E^{k_3}_{k_1,k_2}\big(-\varpi\, t(1-t)\big)dt, (h, g \in \mathbb{R}, \varpi, k_1, k_2, k_3 \in \mathbb{R}^+)$ is the extended modified incomplete beta function.

5. Conclusion

By using three parameters function $E^{k_3}_{k_1,k_2}$ introduced by Prabhakar [1], we have to defined a new modified gamma $\Gamma^{k_1,k_2}_{k_3}$ and beta function B^{k_1,k_2}_{ϖ,k_3}. In their special cases, these generalizations include the extension of gamma and beta functions which were presented in [9]. We have investigated some properties of these generalized functions, most of which are analogous with the classical and other related generalized gamma and beta functions. It is expected that the results obtained in this study will be prove significant in area of statistic, physics, engineering and applied mathematics.

Acknowledgement. The author would like to thank Prof. (Dr.) Chena Ram, Former Head, Department of Mathematics and Statistics, Jai Narain Vyas University, Jodhpur for the careful reading of the manuscript and valuable comments. We also thankful to the reviewers for the valuable comments.

Conflict of interest. The authors declare that they have no conflict of interest.

Authors' Contributions. All the authors contribute equally to this work.

REFERENCES

1. T.R. Prabhakar, A singular integral equation with generalized Mittage-Leffler function in the kernel, *Yokohama Math. J.*, 19 (1971) 7-15.

2. M.A. Chaudhry and S.M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*, Chapman and Hall (CRC Press Company), Boca Raton, London, New York and Washington, D.C., 2001.
3. M.A. Chaudhry, A.Qadir, M.Rafiq and S.M Zubair, Extension of Euler beta function, *Journal of computational and Applied Mathematics*, 78 (1997) 19-32.
4. D.M. Lee, A.K. Rathie, R.K.Parmer and Y.S.Kim, Generalized extended Beta function, hypergeometric and confluent hypergeometric functions, *Honam Mathematical Journal*, 33(2) (2011) 187-206.
5. J.Choi, A.K.Pallie and R.K.Parmer, Extension of extended beta, hypergeometric and confluent hypergeometric functions, *Honam Mathematical Journal*, 36(2) (2014) 357-385.
6. E. Ozergin, M.A. Ozarslan and A. Altin, Extension of Gamma, Beta and hypergeometric function, *Journal of Computational and Applied Mathematics*, 235 (2011) 4601-4610.
7. R.K.Parmer, A new generalization of gamma,beta, hypergeometric and confluent hypergeometric functions, *le Mathematiche,* LXVIII (II) (2013) 33-54.
8. P.Agarwal, J. Choi and S Jain, Extended hypergeometric functions of two and three variables, *Communication of the Korean Mathematical Society*, 30(4) (2015) 403-414.
9. P.I.Pucheta, An new extended beta function, *International Journal of Mathematics and its Applications*, 5(3) (2017) 225-260.
10. M.Chand, H.Hachimi and R.Rani, New extension of beta function and its applications, *International Journal of Mathematical Sciences*, Article ID:6451592 (2018) 25pages.
11. K.S.Gehlot and K.S.Nisar, Extension of two parameter Gamma, Beta functions and its properties, *Application in Applied Mathematics*, 6 (2020) 37-55.
12. E.C.Emmanuel, On analytical review of the gamma functions, *Asian Research Journal of Current Science*, 2(1) (2020) 28-33.
13. M.A.H.Kulip, F.F Moshen and S.S.Barahmah, Further extended Gamma and Beta functions in terms of generalized wright functions, *Electronics Journal of University of Aden for Basic and Applied Sciences*, 1(22) (2020) 78-83.
14. A.Ata and I.O.Kiyma, A study on certain properties of generalized special function defined by Fox-Wright function, *Applied Mathematics and Nonlinear Sciences*, 5(1) (2020) 147-162.
15. F.He, A.Bakhet, M.Abdullah and M.Hidan, On the extended hypergeometric functions and their applications for the derivatives of the extended Jacobi matrix polynomials, *Mathematical Problems in Engineering,* Article ID: 4268361 (2020) 8 pages.
16. R.Sahin and O.Yagci, Fractional calculus of the extended hypergeometric function, *Applied Mathematics and Nonlinear Sciences*, 5(1) (2020) 369-384.
17. K.S.Nisar, D.L.Suthar, A.Agarwal and S.D.Purohit, Fractional calculus operators with Appell function kernels applied to Srivastava polynomials and extended Mittage-Leffler function, *Advance in Difference Equations*, 148 (2020) 14 pages.
18. T.Kim and D.S.Kim, Note on the degenerate gamma function, *Russian Journal of Mathematical Physics*, 27(3) (2020) 352-358.

19. A.Tassaddiq, An application of theory of distributions to the family of λ −generalized gamma function, *Mathematics*, 5(6) (2020) 5839-5858.
20. N.Khan, T.Usman and M.Aman, Extended Beta, hypergeometric and confluent hypergeometric function, *Transactions of National Academy of Science of Azerbaijan Series of Physical-Techinacl and Mathematical Sciences*, 39(1) (2019) 83-97.
21. M.Gayasuddin, N.Khan and M.Ali, A study of the extended Beta, Gauss and confluent hypergeometric functions, *International Journal of Applied Mathematics*, 33(10) (2020) 1-13.
22. D.Baleanu, P.Agarwal, R.K. Parmar, M.M. Alqurashi and S.Salahshour, Extension of the fractional derivative operator of the Riemann-Liouville, *Journal of Nonlinear Science and Applications*, 10 (2017) 2914-2924.
23. P.Agarwal, F.Qi, M.Chand and G.Singh, Some fractional differential equations involving generalized hypergeometric functions, *Journal of Applied Analysis*, 25(1) (2019) 37-44.
24. R.K.Parmar and T.K.Pognay, On the Mathieu -type series for the unified Gauss hypergeometric functions, *Applicable Analysis and Discrete Mathematics*, 14 (2020) 138-149.
25. A.Koraby, M.Ahmed, M.Khaled, E.Ahmed and M.Magdy, Generalization of beta functions in terms of Mittage-Leffler function, *Frontiers in Scientific Research and Technology*, 1 (2020) 81-88.
26. K.Tilahun, H. Tadesses and D.L.Suthar, The extended Bessel -Maitland function and integral operators associated with fractional calculus, *Journal of Mathematics*, Article ID:7582063 (2020) 8 pages.
27. D.L.Suthar, D.Baleanu, S.D.Purohit and E.Ucar, Certain $k-$ fractional operators and images form of k -struve function, *Mathematics*, 5(3) (2020) 1706-1719.
28. D.L.Suthar, A.M.Khan, A.Alaria, S.D.Purohit and J.Singh, Extended Bessel-Maitland function and its properties pertaining of integral transforms and fractional calculus, *Mathematics*, 5(2) (2020) 1400-1414.
29. S. Joshi, E.Mittal and R.M.Pandey, On Euler types interval involving Mittage -Leffler functions, *Boletim da Sociedade Paranaense de Matemática*, 38(2) (2020) 125- 134.
30. N.U.Khan and S.W.Khan, A new extension of the Mittage-Leffler function, *Palestine Journal of Mathematics*, 9(2) (2020) 977-983.
31. M.Saif, A.H.Khan and K.S.Nisar, Integral transform of extended Mittage-Leffler in terms of Fox-Wright function, *Plastine Journal of Mathematics*, 9(1) (2020) 456-463.
32. K.S.Gehlot, Differential equation of $p-k$ Mittage-Leffler function, *Palestine Journal of Mathematics*, 9(2) (2020) 940-944.
33. P.I.Pucheta, An extended of $p-k$ Mittage-Leffler function, *Palestine Journal of Mathematics*, 9(2) (2020) 785-791.
34. U.M.Abubakar and S.R.Kabara, A note on a new extended Gamma and Beta functions and their properties, *IOSR Journal of Mathematics*, 15(5) (2019) 1-6.
35. U.M.Abubakar and S.R.Kabara, Some results on the extension of the extended beta function, *IOSR Journal of Mathematics*, 15(5) (2019) 1-12.
36. E.D.Rainville, *Special Functions*, Macmillan, New York,1960.