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Mean Closed Binding Number and Binding-degree of a Graph

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Abstract. We investigate a refinement of the local, average and closed binding number of a graph in terms of mean closed binding number. Like the closed binding number itself, the mean closed binding number $cbind_m(G)$ of G measures graph vulnerability, which is more sensitive. In this study, we define the parameter and find some bounds on the mean closed binding number of some special graphs. Further, some results have been obtained from graph operations. The closed binding degree of a graph is defined and its value for different graphs is obtained.

Keywords: Local closed binding number, Mean closed Binding Number, Closed Binding Number, Closed binding-degree

AMS Mathematics Subject Classification (2010): 57M15

1. Introduction

Here, we consider finite simple graphs without loops and multiple edges as treated in Harary [9]. One of the most important problems which is solved by the help of graph theory is to design a network model whose resistance for disruptions is more than other networks. Graphs are often used to model real world problems such as in a communication, computer, or spy network. The binding number of a graph provides a measure of the distribution of edges among vertices. It also serves as a vulnerability parameter. Lower value of binding number indicates that graph is more vulnerable for connectedness. Formally it was defined by Woodall [16] in his seminal paper way back in 1947. It is considered to be one of the toughest graph parameters and hence find considerably less number of research papers in seven decades of study. For further details one can refer to [14, 15, 16]. Couple of doctoral theses have been written on binding number of a graph [11, 13]. Binding number and its relation with other graph parameters was considered in [7, 6, 14, 15], etc. Cunningham [8] addressed the computational issues regarding binding number.

Till recently the binding number was considered as a global parameter for a given graph. But in 2019, Aslan [5] considered local binding number at every vertex and eventually the average taken over all vertices to obtain an upper bound for the binding number of a graph. Aslan [5] also obtained average binding number of many classes of graph.

Interesting way of considering closed neighborhoods in place of open neighborhoods has given rise to the closed binding number of a graph as defined by Al-Tobaili [1]. He had also considered different graph products to give binding number [2, 3, 4]. Huilgol et al. [10] have measured the effect of edge operation on binding number of a given graph. Hence this aids in measuring the vulnerability parameter's robustness.

Keeping these as reference, we define local closed binding number of a vertex and mean closed binding number of a graph in the coming sections. First we give some existing results which we will use in proving our results.

2. Basic definitions and results

Here, we consider finite simple graphs without loops and multiple edges as treated in Harary [9]. A finite simple graph G be with the set of vertices $V(G)$ and the set of edges $E(G)$. We list some important results that help us in establishing the our results.

Definition 2.1. *[9] The open neighborhood of a vertex* $u \in V(G)$ *is the set* $N(u) = \{v \in V(G); uv \in E(G)\}$.

The open neighborhood of a set $S \subseteq V(G)$ is $N(S) = \cup_{u\in S} N(u)$.

Definition 2.2. *[9] The closed neighborhood of a vertex* $u \in V(G)$ *is the set* $N[u] = \{v \in V(G); uv \in E(G)\} \cup \{u\}$ *and that of a set* $S \subseteq V(G)$ *is* $N[S] = \cup_{u\in S} N[u]$.

Let $deg(u)$ denote the degree of a vertex u in G. The maximum degree of a graph G is the largest vertex degree of G, denoted $\triangle(G)$, and similarly, the minimum vertex degree is the smallest vertex degree of G, denoted $\delta(G)$.

Woodall [16] first defined the binding number of a graph.

Definition 2.3. *[16] The binding number of a graph* G *is defined as* $bind(G) = min\{\frac{|N(S)|}{|S|}: S \subseteq V(G), S \neq \emptyset$ *and* $N(S) \neq V(G)\}$, *where* $N(S) = \{v/uv \in E(G), \forall u \in S\}$, *is the open neighbourhood set of* S.

Al-Tobaili [1] used the closed neighborhood of a set and defined the closed binding number of a graph G.

Definition 2.4. *[1] The closed binding number of a graph* G *is defined as* $cbind(G) = min\{\frac{|N[S]|}{|S|}: S \subset V(G), S \neq \emptyset$ *and* $N[S] \neq V(G)\}$, *where* $N[S] = \{v; uv \in E(G), \forall u \in S\} \cup \{u\}$, *the closed neighborhood set of* S.

This is in line with the local binding number of a graph and average binding number of a graph defined by Aslan [5].

In [5], for a vertex $v \in V(G)$, the local binding number at a vertex v is given as $bind_v(G) = \min_{v \in F_v(G)}\{\frac{|N(S)|}{|S|}\}$, where $F_v(G) = \{S \subset V(G): v \in S, S \neq \emptyset, N(S) \neq V(G)\}$.

A local binding set of v in G is $S \in F_v(G)$, such that $bind_v(G) = \frac{|N(S)|}{|S|}$.

Clearly $bind(G) = \min_{v \in V(G)}\{bind_v(G)\}$.

Furthermore, the average binding number of G is defined as

$$bind_{av}(G) = \frac{1}{n}\sum_{v \in V(G)} bind_v(G),$$

where n is the number of vertices in G.

Definition 2.5. *[9] The join $G + H$ of two graphs G and H is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv: u \in V(G)$ and $v \in E(H)\}$.*

Proposition 2.1. *[16] If $b(G) \geq c$, then every vertex of G has degree $deg(v) \geq \frac{|G|(c-1)}{c} + \frac{1}{c}$. Thus if G is a graph on n vertices with minimum degree $\delta(G)$, then $b(G) \leq \frac{n-1}{n-\delta(G)}$.*

Proposition 2.2. *[1] If G is a spanning sub graph of H, then $b(G) \leq b(H)$.*

Proposition 2.3. *[16] If $m, n \geq 1$, then $b(K_{m,n}) = min\{\frac{m}{n}, \frac{n}{m}\} + 1$.*

Corollary 2.1. *[1] If G is a graph of order n with minimum degree $\delta(G)$, then $cb(G) \leq \frac{n-1}{n-(\delta(G)+1)}$.*

Theorem 2.1. *[1] If H is a spanning sub graph of G, then $Cb(H) \leq Cb(G)$.*

Theorem 2.2. *[1] $Cb(G) \geq 1$.*

Corollary 2.2. *[1] $1 \leq cb(G) \leq \frac{n-1}{n-(\delta(G)+1)}$, $\delta(G)$ is minimum degree of G.*

Theorem 2.3. *[1] If $m, n \geq 1$, then $cb(K_{m,n}) = min\{\frac{m}{n-1}, \frac{n}{m-1}\} + 1$.*

Theorem 2.4. *[1] If $n \geq 4$, then $cb(C_n) = \frac{n-1}{n-3}$.*

Theorem 2.5. *[1] If $n \geq 5$, then $cb(\overline{C_n}) = \frac{n-1}{2}$.*

Theorem 2.6. *[1] If $n \geq 4$, then $cb(P_n) = \frac{n-1}{n-2}$.*

Theorem 2.7. *[1] If $n \geq 5$, then $cb(\overline{P_n}) = \frac{n-1}{2}$.*

3. Mean closed binding number of a graph

In this section we formally introduce the concept of mean closed binding number of a graph and study it in detail.

Definition 3.1. *For $v \in V(G)$, the local closed binding number of v is $cbind_v(G) = \min_{v \in F_v(G)}\{\frac{|N[S]|}{|S|}\}$ where $F_v(G) = \{S \subset V(G): v \in S, S \neq \emptyset, N[S] \neq V(G)\}$.*

Remark 1. *Clearly $cbind(G) = \min_{v \in V(G)}\{cbind_v(G)\}$. A closed local binding set of v in G is $S \in F_v(G)$, such that $cbind_v(G) = \frac{|N[S]|}{|S|}$.*

Definition 3.2. *The mean closed binding number of G is defined as*

$$cbind_m(G) = \frac{1}{n}\sum_{v \in V(G)} cbind_v(G),$$

where n is the number of vertices in graph G.

Remark 2. *A vertex v in G, $cbind_v(G)$ exists only when the degree of v is strictly less than $n-1$. Further, for a graph G, with its $\gamma(G) = 1$, $cbind_m(G)$ does not exist because there is at least one vertex of degree $n-1$ in G.*

Hence, unless mentioned otherwise, in this paper we consider graphs without full degree vertices.

Example 1.

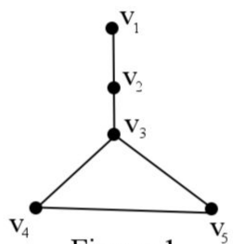

Figure 1

For the graph of Figure 1, we have
$bind(G) = 1$, $cbind(G) = \frac{4}{3}$, $cbind_{v_1} = \frac{3}{2}$,
$cbind_{v_2} = \frac{3}{2}$, $cbind_{v_3} = \frac{4}{3}$, $cbind_{v_4} = \frac{4}{3}$ and $cbind_{v_5} = \frac{4}{3}$.
It follows that $cbind_m(G) = \frac{1}{5}(\frac{3}{2} + \frac{3}{2} + \frac{4}{3} + \frac{4}{3} + \frac{4}{3}) = \frac{7}{5}$.

Example 2.

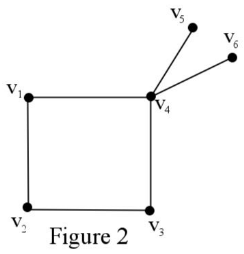

Figure 2

For the graph of Figure 2, we have $bind(G) = \frac{1}{2}$, $cbind(G) = \frac{5}{4}$
$cbind_{v_1} = \frac{5}{4}$, $cbind_{v_2} = \frac{5}{4}$, $cbind_{v_3} = \frac{5}{4}$, $cbind_{v_4} = \frac{5}{3}$, $cbind_{v_5} = \frac{5}{4}$
and $cbind_{v_6} = \frac{5}{4}$.

It follows that $cbind_m(G) = \frac{1}{5}(\frac{5}{4} + \frac{5}{4} + \frac{5}{4} + \frac{5}{3} + \frac{5}{4} + \frac{5}{4}) = \frac{95}{72}$.

4. Bounds of mean closed binding number

Here, we find some bounds on mean closed binding number for some graphs. The related theorems of mean closed binding number and other graph parameters are provided as follows.

Theorem 4.1. *If G is a graph of order n with minimum degree $\delta(G)$, then*

$$cbind_m(G) \geq \frac{\delta(G)+1}{n-(\delta(G)+1)}.$$

Proof: For $v \in V(G)$ and S_v be a local closed binding set at v. Certainly $|N[S_v]| \geq deg(v) + 1 \geq \delta(G) + 1$. Since $N[S_v] \neq V(G)$, each S_v omits all closed neighbors, as each closed neighborhood sets to miss some vertices. The missing vertices are of smallest degree, say w and observe that $|S_v| \leq n - (deg(w) + 1) \leq n - (\delta(G) + 1)$.

Therefore

$$cbind_v(G) = \frac{|N[S_v]|}{|S_v|} \geq \frac{\delta(G)+1}{n-(\delta(G)+1)}.$$

Hence

$$cbind_m(G) = \frac{1}{n}\sum_{v\in V(G)} cbind_v(G) \geq \frac{\delta(G)+1}{n-(\delta(G)+1)}.$$

Theorem 4.2. *If G_2 is a spanning subgraph of G_1, then $cbind_m(G_2) \leq cbind_m(G_1)$.*
Proof: Let $v \in V(G_1)$ with a local closed binding set $S'_v \in F_v(G_1)$, observe that $N_{G_1}[S'_v] = N[S'_v] \cap V(G_1)$ and $N_{G_2}[S'_v] = N[S'_v] \cap V(G_2)$. Then $N_{G_2}[S'_v] \leq N_{G_1}[S'_v]$ and $|S'_v|$ in G_2 is at least $|S'_v|$ in G_1 ($|S'_v|_{G_2} \geq |S'_v|_{G_1}$). Therefore,

$$cbind_v(G_2) = \frac{|N_{G_2}[S'_v]|}{|S'_v|} \leq \frac{|N_{G_1}[S'_v]|}{|S'_v|} = cbind_v(G_1).$$

Hence,

$$cbind_m(G_2) \leq cbind_m(G_1).$$

Theorem 4.3. *If G is a graph of order n, then $cbind(G) \leq cbind_m(G)$.*
Proof: For $v \in V(G)$, $cbind(G) \leq cbind_v(G)$, thus

$$cbind(G) \leq \frac{1}{n}\sum_{v\in V(G)} cbind_v(G) = cbind_m(G).$$

Remark 3. *The closed binding number varies form* $1 \leq cbind(G) \leq n-1$. *The lower bound attained in a graph with two or more components, which are complete graphs and upper bound is attained by any* $(n-2)$*-regular graph.*

5. Mean closed binding number of certain classes of graphs

We present some results on the mean closed binding number of some standard graphs.

Theorem 5.1. *If* P_n *is a path,* $\forall\ n \geq 4$, *then* $cbind_m(P_n) = \frac{n-1}{n-2}$.

Proof: For $v \in V(P_n)$, $\forall\ n \geq 4$ and $S_v \in F_v(P_n)$ and S_v contains all vertices of P_n except a pendant vertex and its adjacent vertex. It is clear that $N[S_v]$ consists of all vertices of P_n except one pendant vertex. This implies that, $cbind_v(P_n) \leq \frac{n-1}{n-2}$.

Thus by definition,

$$cbind_m(P_n) \leq \frac{n-1}{n-2}. \tag{5.1}$$

From Theorem 2.6 [1] and Theorem 4.3, it follows that,

$$cbind_m(P_n) \geq \frac{n-1}{n-2}. \tag{5.2}$$

From (5.1) and (5.2), the result holds.

Next result deals with the complement of paths.

Theorem 5.2. *If* $n \geq 5$, *then* $cbind_m(\overline{P_n}) = \frac{n-1}{2}$.

Proof: For $v \in V(\overline{P_n})$ $\forall\ n \geq 4$ and $S_v \in F_v(\overline{P_n})$, observe that S_v consists of only two adjacent vertices that means $|S_v| = 2$ and $|N[S_v]| = n-1$. Thus $cbind_v(\overline{P_n}) \leq \frac{n-1}{2}$. Therefore,

$$cbind_m(\overline{P_n}) \leq \frac{n-1}{2}. \tag{5.3}$$

From Theorem 2.7 [1] and Theorem 4.3, we get,

$$cbind_m(\overline{P_n}) \geq \frac{n-1}{2}. \tag{5.4}$$

From (5.3) and (5.4), the result holds good.

Theorem 5.3. *If* C_n *is a cycle,* $forall\ n \geq 4$, *then* $cbind_m(C_n) = \frac{n-1}{n-3}$.

Proof: For $v \in V(C_n)$, $\forall\ n \geq 4$ and $S_v \in F_v(C_n)$. Let $|S_v| = r$ we get $r \leq n - (\delta(G)+1) = n-3$ and $|N[S_v]| > r+2$. Thus $\frac{|N[S_v]|}{|S_v|} > \frac{r+2}{r}$ is a decreasing function of r that, the minimum value of $r = n-3$. Hence, $cbind_v(C_n) \leq \frac{n-1}{n-3}$, as there is a closed binding set S'_v of C_n such that $|S'_v| = n-3$ and $|N[S'_v]| = n-1$.

Therefore,

$$cbind_m(C_n) \leq \frac{n-1}{n-3}. \tag{5.5}$$

form Theorem 2.4 [1] and Theorem 4.3, we see that,

$$cbind_m(C_n) \geq \frac{n-1}{n-3}. \tag{5.6}$$

From (5.5) and (5.6), the result holds.

In the following result we consider the complement of a cycle.

Theorem 5.4. *If* $n \geq 6$, *then* $cbind_m(\overline{C_n}) = \frac{n-1}{2}$.

Proof: For $v \in V(\overline{C_n})$ $\forall$ $n \geq 4$ and $S_v \in F_v(\overline{C_n})$, observe that S_v consists of only two adjacent vertices that means $|S_v| = 2$ and $|N[S_v]| = n - 1$. Thus $cbind_v(\overline{C_n}) \leq \frac{n-1}{2}$. Therefore,

$$cbind_m(\bar{C}_n) \leq \frac{n-1}{2}. \tag{5.7}$$

From Theorem 2.5 [1] and Theorem 4.3, we get,

$$cbind_m(\bar{C}_n) \geq \frac{n-1}{2}. \tag{5.8}$$

From (5.7) and (5.8), the result holds good.

Theorem 5.5. *If* $K_{m,n}$ *is a complete bipartite graph of order* $m, n \geq 3$, *then*

$$cbind_m(K_{m,n}) \leq \frac{2(m^2n+n^2m-2mn)-(m^2+n^2)+m+n}{m^2n+n^2m-(m^2+n^2)-2mn+m+n}.$$

Proof: Let G_1 and G_2 be partite sets of $K_{m,n}$ and $V(K_{m,n}) = V(G_1) \cup V(G_2)$ be the vertex set of $K_{m,n}$, where $V(G_1)$ contains m vertices having degree n and the set $V(G_2)$ contains n vertices having degree m. For $v \in V(K_{m,n})$ and $S_v \in F(K_{m,n})$, such that S_v is a closed binding set of $K_{m,n}$. Since $V(G_1)$ and $V(G_2)$ are independent sets of vertices, $N[S_v] \neq V(K_{m,n})$. S_{v_i} must contain vertices from $V(G_1)$ or $V(G_2)$, then clearly the ratio $\frac{|N[S_{v_i}]|}{|S_{v_i}|}$ is minimized, where $S_{v_i} = V(G_1) - \{v_j\}$ and $i \neq j$ for $i, j = 1,2,\cdots,m$ or $i, j = 1,2,\cdots,n$. So, $N[S_{v_i}] = V(G_1) \cup V(G_2) - 1$, thus $|N[S_{v_i}]| = m + n - 1$ and $|S_{v_i}| = m - 1$ or $n - 1$. Then $cbind_{v_i}(K_{m,n}) \leq \frac{m+n-1}{m-1}$ or $\frac{m+n-1}{n-1}$. Therefore

$$cbind_m(K_{m,n}) \leq \frac{2(m^2n+n^2m-2mn)-(m^2+n^2)+m+n}{m^2n+n^2m-(m^2+n^2)-2mn+m+n}.$$

Remark 4. *The complement of* $K_{m,n}$ *is divided into two components which are complete graphs. Therefore* $cbind_m(\overline{K_{m,n}}) = 1$.

Definition 5.1. *The double star* $S_{m,n}$ *is a tree with diameter* 3 *and central vertices of degree* m *and* n *respectively.*

Theorem 5.6. *If* $S_{m,n}$ *is a double star,* $forall$ $m, n \geq 2$, *then*

$$cbind_m(S_{m,n}) \leq \frac{m+n-1}{m+n-2}.$$

Proof: For $v \in V(S_{m,n})$, $m, n \geq 2$ and $S_v \in F_v(S_{m,n})$ and S_v consists of all vertices of $S_{m,n}$ except two adjacent vertices with one vertex of degree one and other vertex of degree m or n. Then $N[S_v]$ consists of all vertices of $S_{m,n}$ except only one vertex of degree one, implying that $cbind_v(S_{m,n}) \leq \frac{m+n-1}{m+n-2}$.

Thus by definition

$$cbind_m(S_{m,n}) \leq \frac{m+n-1}{m+n-2}.$$

Theorem 5.7. *If $\overline{S_{m,n}}$ is the complement of a double star of order $m,n \geq 2$, then*

$$cbind_m(\overline{S_{m,n}}) \leq \frac{m^2+n^2+2mn-m-n}{m^2n+mn^2}.$$

Proof: For $v \in V(\overline{S_{m,n}})$ $\forall$ $m,n \geq 2$ and $S_{v_i} \in F_{v_i}(\overline{S_{m,n}})$, observe that $|N[S_{v_i}]| > |S_{v_i}|$ with equality holds whenever S_{v_i} for $1 \leq i \leq m+n$ consists of only two sets containing m and n vertices respectively and $N[S_{v_i}]$ consists of all vertices of $\overline{S_{m,n}}$ except only one vertex of degree $n-1$ and $m-1$ respectively. Therefore $cbind_{v_i} \leq \frac{m+n-1}{m}$ or $\frac{m+n-1}{n}$.

Thus by definition $cbind_m(\overline{S_{m,n}}) \leq \frac{m^2+n^2+2mn-m-n}{m^2n+m^2n}$.

Next we will consider the Jahangir graphs. First of all we will define them here.

Definition 5.2. *Jahangir graphs $J_{n,m}$ for $m \geq 3$, is a graph on $nm+1$ vertices consisting of a cycle $C_{m,n}$ with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}.*

Theorem 5.8. *If $J_{2,n}$ be a Jahangir graph of order $n \geq 3$, then*

$$cbind_m(J_{2,n}) \leq \frac{n}{n-1}.$$

Proof: For $v \in V(J_{2,n})$, $\forall$ $n \geq 3$ and $S_v \in F_v(J_{2,n})$ and whenever S_v consists of all vertices of $J_{2,n}$ except three adjacent vertices with one vertex of degree 2 and other vertex of degree 3. Then $N[S_v]$ consists of all vertices of $J_{2,n}$ except only one vertex of degree two. Therefore $cbind_v(J_{2,n}) \leq \frac{2n}{2n-2} = \frac{n}{n-1}$.

Thus by definition

$$cbind_m(J_{2,n}) \leq \frac{n}{n-1}.$$

Theorem 5.9. *If $\overline{J_{2,n}}$ is the complement of Jahangir graph of order $n \geq 3$, then*

$$cbind_m(\overline{J_{2,n}}) \leq \frac{n^2+8n}{3(2n+1)}.$$

Proof: For $v \in V(\overline{J_{2,n}})$, $\forall$ $n \geq 3$ and $S_{v_i} \in F_{v_i}(\overline{J_{2,n}})$, observe that $|N[S_{v_i}]| > |S_{v_i}|$ with equality holds whenever S_{v_i} for $1 \leq i \leq 2n+1$ consists of $2n+1$ vertices. In n sets containing n vertices and $n+1$ sets containing only 3 vertices, respectively, then $N[S_{v_i}]$ consists of all vertices of $\overline{J_{2,n}}$ except only one vertex. Therefore $cbind_{v_i}(\overline{J_{2,n}}) \leq \frac{2n}{n} = 2$ or $\frac{2n}{3}$.

Thus by definition

$$cbind_m(\overline{J_{2,n}}) \leq \frac{1}{2n+1}\left(2n + (n+1)\frac{2n}{3}\right) = \frac{2(n^2+4n)}{3(2n+1)}.$$

6. Join of graphs

We provide some results of the mean closed binding number of join of graphs. First we define the join of two graphs.

Definition 6.1. *[9] The join $G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ is the graph consisting of G_1, G_2 and all edges joining $V(G_1)$ and $V(G_2)$.*

Theorem 6.1. *Let G_1 and G_2 be two connected graphs of order m and n, respectively. Then $cbind_m(G_1 + G_2) \geq \frac{m\times cbind_m(G_1)+n\times cbind_m(G_2)}{m+n}$.*

Proof: For $v \in V(G_1 + G_2)$ and $S_v \in F_v(G_1 + G_2)$, observe that $N_{G_1}[S_v] = N[S_v] \cap V(G_1)$ and $N_{G_2}[S_v] = N[S_v] \cap V(G_2)$. Since every vertex of G_1 is connected to all vertices of G_2 and vice versa, we get $N[S_v] = N_{G_1}[S_v] \cup V(G_2)$ or $N[S_v] = N_{G_2}[S_v] \cup V(G_1)$, since $N[S_v] \neq V(G_1 + G_2)$. For $v \in V(G_1)$ implies $S_v \subseteq V(G_1)$ or $v \in V(G_2)$ implies $S_v \subseteq V(G_2)$. For $v \in V(G_1)$, let S'_v be its local closed binding set in G_1. Since $S_v \subseteq V(G_1)$, $S_v \in F_v(G_1 + G_2) \cap F_v(G_1)$. Thus

$$\frac{|N[S_v]|}{|S_v|} = \frac{|N_{G_1}[S_v] \cup V(G_2)|}{|S_v|} = \frac{|N_{G_1}[S_v] + V(G_2)|}{|S_v|}$$
$$\geq \frac{|N_{G_1}[S'_v]|}{|S'_v|} + \frac{|V(G_2)|}{|S_v|}$$
$$\geq cbind_v(G_1) + \frac{|V(G_2)|}{|S_v|}$$
$$\geq cbind_v(G_1).$$

Then

$$m \times cbind_v(G_1 + G_2) \geq m \times cbind_v(G_1). \qquad (6.1)$$

Similarly for $v \in V(G_2)$,

$$n \times cbind_v(G_1 + G_2) \geq n \times cbind_v(G_2). \qquad (6.2)$$

Hence by (6.1) and (6.2) we get

$$cbind_v(G_1 + G_2) \geq \frac{m\times cbind_v(G_1)+n\times cbind_v(G_2)}{m+n}.$$

Therefore by definition,

$$cbind_m(G_1 + G_2) \geq \frac{m\times cbind_m(G_1)+n\times cbind_m(G_2)}{m+n}.$$

The above equality holds good for following results:

Theorem 6.2. *For m and n are two positive integers, then*

$$cbind_m(P_m + P_n) = \frac{2(m^2n+mn^2-3mn-m^2-n^2+m+n)}{m^2n+mn^2-2n^2-2m^2-4mn+4m+4n}.$$

Proof: Let $v \in V(P_m + P_n)$ and $S_v \in F_v(P_m + P_n)$. By Lemma 4.1, if $S_v \cap V(P_m) \neq \emptyset$ and $S_v \cap V(P_n) \neq \emptyset$, then $N[S_v] = V(P_m + P_n)$ a contradiction. Hence either $S_v \subseteq V(P_m)$ or $S_v \subseteq V(P_n)$.

If $v \in V(P_m)$ then $|S_v| = m - 2$, otherwise $N[S_v] = V(P_m + P_n)$. Thus $cbind_v(P_m + P_n) = \frac{m+n-1}{m-2}$ and $\sum_{v\in V(P_m)} cbind_v(P_m + P_n) = m\frac{m+n-1}{m-2}$.

Again if $v \in V(P_n)$ then $|S_v| = n - 2$, otherwise $N[S_v] = V(P_m + P_n)$. Thus $cbind_v(P_m + P_n) = \frac{m+n-1}{n-2}$ and $\sum_{v\in V(P_n)} cbind_v(P_m + P_n) = n\frac{m+n-1}{n-2}$. Hence

$$cbind_m(P_m + P_n) = \frac{2(m^2n+mn^2-3mn-m^2-n^2+m+n)}{m^2n+mn^2-2n^2-2m^2-4mn+4m+4n}.$$

Theorem 6.3. *For m and n are two positive integers, then*

$$cbind_m(P_m + C_n) = \frac{2m^2n+2mn^2-7mn-3m^2-2n^2+3m+2n}{m^2n+mn^2-2n^2-3m^2-5mn+6m+6n}.$$

Proof: For $v \in V(P_m + C_n)$ and $S_v \in F_v(P_m + C_n)$ by Lemma 4.1, if $S_v \cap V(P_m) \neq \emptyset$ and $S_v \cap V(C_n) \neq \emptyset$. Then $N[S_v] = V(P_m + C_n)$ a contradiction. Hence either $S_v \subseteq V(P_m)$ or $S_v \subseteq V(C_n)$.

If $v \in V(P_m)$ then $|S_v| = m - 2$, otherwise $N[S_v] = V(P_m + C_n)$. Thus $cbind_v(P_m + C_n) = \frac{m+n-1}{m-2}$ and $\sum_{v \in V(P_m)} cbind_v(P_m + C_n) = m\frac{m+n-1}{m-2}$.

Again if $v \in V(C_n)$ then $|S_v| = n - 3$, otherwise $N[S_v] = V(P_m + C_n)$. Thus $cbind_v(P_m + C_n) = \frac{m+n-1}{n-3}$ and $\sum_{v \in V(C_n)} cbind_v(P_m + C_n) = n\frac{m+n-1}{n-3}$. Hence

$$cbind_m(P_m + C_n) = \frac{2m^2n+2mn^2-7mn-3m^2-2n^2+3m+2n}{m^2n+mn^2-2n^2-3m^2-5mn+6m+6n}.$$

Theorem 6.4. *For m and n are two positive integers, then*

$$cbind_m(C_m + C_n) = \frac{2m^2n+2mn^2-8mn-3m^2-3n^2+3m+3n}{m^2n+mn^2-3n^2-3m^2-6mn+9m+9n}.$$

Proof: For $v \in V(C_m + C_n)$ and $S_v \in F_v(C_m + C_n)$ by Lemma 4.1, if $S_v \cap V(C_m) \neq \emptyset$ and $S_v \cap V(C_n) \neq \emptyset$. Then $N[S_v] = V(C_m + C_n)$ a contradiction. Hence either $S_v \subseteq V(C_m)$ or $S_v \subseteq V(C_n)$.

If $v \in V(C_m)$ then $|S_v| = m - 3$, otherwise $N[S_v] = V(C_m + C_n)$. Thus $cbind_v(C_m + C_n) = \frac{m+n-1}{m-3}$ and $\sum_{v \in V(C_m)} cbind_v(C_m + C_n) = m\frac{m+n-1}{m-3}$.

Again if $v \in V(C_n)$ then $|S_v| = n - 3$, otherwise $N[S_v] = V(C_m + C_n)$. Thus $cbind_v(C_m + C_n) = \frac{m+n-1}{n-3}$ and $\sum_{v \in V(C_n)} cbind_v(C_m + C_n) = n\frac{m+n-1}{n-3}$. Hence

$$cbind_m(C_m + C_n) = \frac{2m^2n+2mn^2-8mn-3m^2-3n^2+3m+3n}{m^2n+mn^2-3n^2-3m^2-6mn+9m+9n}.$$

Lemma 6.1. *Let G_1 and G_2 be two connected graphs. If $v \in V(G_1)$ or $v \in V(G_2)$ and S_v is local closed binding set of $G_1 + G_2$, then S_v is either a local closed binding set of G_1 or G_2.*

Proof: Suppose that $v \in V(G_1)$. Let $N_{G_1}[S_v] = N[S_v] \cap V(G_1)$. Since all vertices of G are adjacent to all vertices of G_2, $N[S_v] = N_{G_1}[S_v] \cup V(G_2)$ which implies that $S_v \subset V(G_1)$. Hence S_v is a local binding set of G_1. Similarly, if $v \in V(G_2)$, then S_v is a local binding set of G_2.

7. Closed binding degree of G

The closed binding number and local closed binding number of a graph give layered measures of the vulnerability of a graph. This is motivated by Huilgol et al. [10] to define *closed binding degree* of a graph. We first define the parameter and then determine it for different class of graphs.

Definition 7.1. *The Closed binding degree of G, denoted by $b_c^d(G)$, is defined as*

$$b_c^d(G) = \sum_{i=1}^{n} deg(v_i) cbind_{v_i}(G).$$

Illustration 7.1. For the graph of Figure 1, we have $|V(G)| = 5$ and $|E(G)| = 5$. Note that $bind(G) = 1$, $cbind(G) = \frac{4}{3}$, $deg(v_1) = 1$, $cbind_{v_1} = \frac{3}{2}$, $deg(v_2) = 2$, $cbind_{v_2} = \frac{3}{2}$, $deg(v_3) = 3$ $cbind_{v_3} = \frac{4}{3}$, $deg(v_4) = 2$, $cbind_{v_4} = \frac{4}{3}$, $deg(v_5) = 2$ and $cbind_{v_5} = \frac{4}{3}$. It follows that,

$$b_c^d(G) = 1 \times \frac{3}{2} + 2 \times \frac{3}{2} + 3 \times \frac{4}{5} + 2 \times \frac{4}{3} + 2 \times \frac{4}{3} = \frac{83}{6}.$$

Proposition 7.1. *The closed binding degree of cycles, paths, complete bipartite graphs, double star, Jahangir graph, complement of cycles, complement of paths, complement of double star, complement of Jahangir graph is given as follows:*

- For any path P_n with $n \geq 4$ vertices,
$$b_c^d(P_n) = \frac{(n-1)^2}{n-2}.$$

- For any complement of path $\overline{P_n}$ with $n \geq 5$ vertices,
$$b_c^d(\overline{P_n}) = \frac{(n-1)^2(n-2)}{2}.$$

- For any cycle C_n with $n \geq 4$ vertices,
$$b_c^d(C_n) = \frac{2n(n-1)}{n-3}.$$

- For any complement of cycle $\overline{C_n}$ with $n \geq 6$ vertices,
$$b_c^d(\overline{C_n}) = \frac{n(n-1)(n-3)}{2}.$$

- For any complete bipartite graph $K_{m,n}$,
$$b_c^d(K_{m,n}) \leq \frac{mn((m+n)^2-3(m+n)+2}{mn-m-n+1}.$$

- For $m, n \geq 3$, the binding degree of a double star $S_{m,n}$ is,
$$b_c^d(S_{m,n}) \leq 2\frac{(m+n-1)^2}{m+n-2}.$$

- For $m, n \geq 4$, the binding degree of a complement of double star $\overline{S_{m,n}}$ is,
$$b_c^d(\overline{S_{m,n}}) \leq 2(m+n-1)\frac{2mn-(m+n)}{mn}.$$

- For $n \geq 3$, the binding degree of a Jahangir graph $J_{2,n}$ is,
$$b_c^d(J_{2,n}) \leq \frac{6n^2}{n-1}.$$

- For $n \geq 3$, the binding degree of a complement of Jahangir graph $\overline{J_{2,n}}$ is,
$$b_c^d(\overline{J_{2,n}}) \leq \frac{2n(2n^2+5n-9)}{3}.$$

Proof: (i) Let P_n be a path with $n \geq 4$ vertices. If the vertices of P_n are labeled as $v_1, v_2, \cdots, v_n$ then we know that $deg(v_1) = deg(v_n) = 1$ and $deg(v_2) = \cdots =$

$deg(v_{n-1}) = 2$. For $v_i \in V(P_n)$, the local binding number of v_i with $1 \le i \le n$ is $cbind_{v_i}(P_n) = \frac{n-1}{n-2}$. Therefore $b_c^d(G) = 2 \times 1 \times \frac{n-1}{n-2} + (n-2) \times 2 \times \frac{n-1}{n-2} = \frac{2(n-1)^2}{n-2}$.

(ii) Let $\overline{P_n}$ be the complement of path with $n \ge 5$ vertices. If the vertices of $\overline{P_n}$ are labeled as $v_1, v_2, \cdots, v_n$ then we know that $deg(v_1) = deg(v_n) = n-2$ and $deg(v_2) = \cdots = deg(v_{n-1}) = n-3$. For $v_i \in V(\overline{P_n})$, the local binding number of v_i with $1 \le i \le n$ is $cbind_{v_i}(\overline{P_n}) = \frac{n-1}{2}$. Therefore $b_c^d(\overline{P_n}) = 2 \times (n-2) \times \frac{n-1}{2} + (n-2) \times (n-3) \times \frac{n-1}{2} = \frac{(n-1)^2(n-2)}{2}$.

(iii) Let C_n be a cycle with $n \ge 3$ vertices. We know that C_n is a self-centered, regular graph of regularity 2. From [1], we know that $cbind(C_n) = \frac{n-1}{n-3}$. Therefore $b_c^d(C_n) = \frac{2n(n-1)}{n-3}$.

(iv) Let $\overline{C_n}$ be a cycle with $n \ge 6$ vertices. We know that $\overline{C_n}$ is a self-centered, regular graph of regularity $n-3$. From [1], we know that $cbind(\overline{C_n}) = \frac{n-1}{2}$. Therefore $b_c^d(\overline{C_n}) = \frac{n(n-1)(n-3)}{2}$.

(v) Let $K_{m,n}$ be a complete bipartite graph with $m, n \ge 3$ vertices. If the vertices of $K_{m,n}$ are labeled as $u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n$. For $u_i \in V(K_{m,n})$, the local closed binding number of u_i for $1 \le i \le m$, is

$cbind_{u_i}(K_{m,n}) \le \frac{m+n-1}{m-1}$, $deg(u_i) = n$ and the local binding number of u_j for $1 \le j \le n$, is $bind_{v_j}(K_{m,n}) \le \frac{m+n-1}{n-1}$ and $deg(v_i) = m$. Therefore $b_c^d(K_{m,n}) \le mn\frac{m+n-1}{m-1} + mn\frac{m+n-1}{n-1} = \frac{mn((m+n)^2-3(m+n)+2}{mn-m-n+1}$.

(vi) Let $S_{m,n}$ be a double star with $m, n \ge 3$ vertices. If the vertices of $S_{m,n}$ are labeled as $u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n$. For $u_i \in V(S_{m,n})$, the local closed binding number of u_i for $1 \le i \le m-1$, is $cbind_{u_i}(S_{m,n}) \le \frac{m+n-1}{m+n-2}$, $deg(u_i) = 1$, $cbind_{u_m}(S_{m,n}) \le \frac{m+n-1}{m+n-2}$, $deg(u_m) = m$ and the local binding number of v_j for $1 \le j \le n-1$, is $cbind_{v_j}(S_{m,n}) \le \frac{m+n-1}{m+n-2}$, $deg(v_j) = 1$, $deg(v_j) = n$ and $cbind_{v_j}(S_{m,n}) \le \frac{m+n-1}{m+n-2}$. Therefore $b_c^d(S_{m,n}) \le m\frac{m+n-1}{m+n-2} + n\frac{m+n-1}{m+n-2} + (m+n-2)\frac{m+n-1}{m+n-2} = 2\frac{(m+n-1)^2}{m+n-2}$.

(vii) Let $\overline{S_{m,n}}$ be a complement of double star with $m, n \ge 3$ vertices. If the vertices of $\overline{S_{m,n}}$ are labeled as $u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n$. For $u_i \in V(\overline{S_{m,n}})$, the local closed binding number of u_i for $1 \le i \le m-1$, is $cbind_{u_i}(\overline{S_{m,n}}) \le \frac{m+n-1}{m}$, $deg(u_i) = 1$, $cbind_{u_m}(\overline{S_{m,n}}) \le \frac{m+n-1}{n}$, $deg(u_m) = n-1$ and the local binding number of v_j for $1 \le j \le n-1$, is $cbind_{v_j}(S_{m,n}) \le \frac{m+n-1}{n}$, $deg(v_j) = 1$, $deg(v_n) = m-1$ and $cbind_{v_j}(\overline{S_{m,n}}) \le \frac{m+n-1}{m}$. Therefore $b_c^d(\overline{S_{m,n}}) \le m-1\frac{m+n-1}{m} + n-1\frac{m+n-1}{n} + n-1\frac{m+n-1}{n} + m-1\frac{m+n-1}{m} = 2(m+n-1)\frac{2mn-(m+n)}{mn}$.

(viii) Let $J_{2,n}$ be a Jahangir graph with $n \geq 3$ vertices. If the vertices of $J_{2,n}$ are labeled as $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n, v_{n+1}$. For $u_i \in V(J_{2,n})$, the local closed binding number of u_i for $1 \leq i \leq n$, is $cbind_{u_i}(J_{2,n}) \leq \frac{n}{n-1}$, $deg(u_i) = 2$ and the local binding number of v_j for $1 \leq j \leq n$, is $cbind_{v_j}(J_{2,n}) \leq \frac{n}{n-1}$, $deg(v_j) = 3$, $deg(v_{n+1}) = n$ and $cbind_{v_{n+1}}(J_{2,n}) \leq \frac{n}{n-1}$.

Therefore $b_c^d(J_{2,n}) \leq n\frac{n}{n-1} + 2n\frac{n}{n-1} + 3n\frac{n}{n-1} = \frac{6n^2}{n-1}$.

(ix) Let $\overline{J_{2,n}}$ be a Jahangir graph with $n \geq 3$ vertices. If the vertices of $\overline{J_{2,n}}$ are labeled as $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n, v_{n+1}$. For $u_i \in V(\overline{J_{2,n}})$, the local closed binding number of u_i for $1 \leq i \leq n$, is $cbind_{u_i}(\overline{J_{2,n}}) \leq \frac{2n}{3}$, $deg(u_i) = 2(n-1)$ and the local binding number of v_j for $1 \leq j \leq n$, is $cbind_{v_j}(\overline{J_{2,n}}) \leq 2$, $deg(v_j) = 2n-3$, $deg(v_{n+1}) = n$ and $cbind_{v_{n+1}}(\overline{J_{2,n}}) \leq \frac{2n}{3}$. Therefore $b_c^d(\overline{J_{2,n}}) \leq n\frac{2n}{3} + 2n(n-1)\frac{2n}{3} + n(2n-3)2 = 2n\frac{2n^2+5n-9}{3}$.

8. Conclusion

In this paper, we have introduced the concept of mean closed binding number and degree of a graph. Many bounds have been established. For many standard classes of graphs we have given exact value. Similar study can be extended for other classes of graphs and more general bounds can be obtained in terms of other graph parameters. We conclude this paper by citing a conjecture by Aslan [1] and proposing a conjecture. Aslan [1] had conjectured a result relating $cbind(G)$ and $bind(G)$.

Conjecture [1]: Let G be a graph other then K_n. Then $cbind(G) \geq bind(G)$.
In [1] Aslan had proved that the conjecture holds good in case of graphs $K_{m,n}$, rK_2, $cp(r)$, W_n, C_n, $\overline{C_n}$, P_n, $\overline{P_n}$ and disconnected graphs.

On similar lines we also conjecture the following on mean closed binding number and average binding number of a graph.

Conjecture: For any graph G, without full degree vertices, $cbind_m(G) \geq bind_{av}(G)$.
We have shown that conjecture holds good in case of P_n, $\overline{P_n}$, C_n, $\overline{C_n}$ and $K_{m,n}$ (Theorem 5.1 to 5.5 respectively). At this juncture we strongly believe that the conjecture is true, but proof seems elusive.

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