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# Mathematical Modelling of Drug Abuse and it's Effect in the Society

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***Abstract.*** Drug abuse remains to be the global burden causing a large number of death and disability, it is now termed as a significant threat to public health for both developed and developing countries. This work presents a mathematical model as a new approach towards understanding and controlling drug abuse in Tanzania. Using next-generation matrix method an epidemic threshold value, $R_0$ is computed and used to establish condition for the existence and stability of stationary points. Using numerical simulation the dynamical behavior of the model is explored and the result shows the significant contribution by the rate of adequate contact between the susceptible individual and the drug user, and the rate of recovery of drug user after rehabilitation as the major factors or players that dictate the dynamics of the drug user population in the society. The model is finally analyzed to study the dynamical behavior of the drug abuse when control is applied, the result shows the drug users are minimized significantly. The result signifies the importance of early control of the drug abuse problem through the establishment of strict laws that will narrow the possibility of this bad practice in societies.

***Keywords:*** Drugs, drug abuse, mathematical model, basic reproductive number, equilibrium points

***AMS Mathematics Subject Classification (2010):*** 92B05

## 1. Introduction

Drug(s) stand for any chemical substance which when used without the directive of the physician affects living organism. These substance(s) have got a variety of use but mostly may be used for the treatment of various health problem of a human being [2,12]. The term drug abuse may be defined as the unlawful, illegitimate, prohibited, illegal, inappropriate, extreme or excessive use of drugs. Teens are increasingly engaging in prescribed to relieve several pain and stimulant medication which treats condition like attention deficit disorder and narcolepsy World health organization (2011).

The most common drugs that are used for drug abused includes alcohol, amphetamines, nicotine, alcohol, barbiturates, Cannabis, cocaine, methaqualone, opium

alkaloids, synthetic opioids and other the kind. Drug abuse if are not controlled at the earliest stage may lead to significant effects on society and to well-being of the user [12]. The excessive use of the drug may result in addiction which may also be termed as dependence or addiction of a particular drug. Drug addiction affects the psychological, social and physical well-being of the human being and the injurious effect on the whole community[15].

Drug users, use different ways to administer drugs depending on the nature of the drugs one is using. The most popular ways of drug administration includes snorting, smoking and oral ingestion. The mode in which drug is being administered determine level of drug uptake in the blood and the related consequences [17].

Africa can not separate itself from the whole world, this means since other continents are affected by the drug abuse problem Africa is also affected significantly. Different studies justifies the use of some drugs like alcohol with poverty, this position Africa onto more vulnerable situation to be affected by drug abuse problem even more than other developed continents. Many countries in Africa like South Africa and Nigeria have portrayed a worse senatorial as the level of drug use in these countries is almost the peak of drug abuse in Africa and many other countries around the world [13].

Tanzania is also among the African countries that are affected by drug abuse. The problem has lead to the number of effect to the society in particular youth who are the main workers in different Tanzania's society. Tanzania recorded a high peak of drug trafficking in the years between 1990 and 1995. The government has taken different measure like severe punishment to the drug dealers in order to eradicate the problem but still the number of drug users is increasing in societies.

The nation drug report shows the seasonal trend of drug use from 2001 to 2005. In 2005 Tanzania recorded a very significant drop of the use of there was decrease of use of cocaine of about 1.5 kg per year compared to the 7.5kg per year recorded in 2001 [7]

It is evident that presence of drug use have attracted many researcher to investigate and search and/or suggest solution for the problem. Regardless of the extensive work about the same but still the problem still persist and the major factors influencing its existence are still unknown. This justifies the need of the study that will thoroughly study the dynamics of drug abuse in the society and identify the factors that dictate its dynamics for a proper control plan.

This study intend to formulate a mathematical model system and analyze it to study the dynamics of drug abuse in the society and suggest the control mechanism. The model is analyzed to establish conditions for existence and extension of the drug abuse problem in the society, the local and global stability of stationary points and the systemâ€™s dynamical behavior through numerical simulation.

## 2. Model development

### 2.1. Model description

In this drug use model, we consider drug user population dynamics in the society, we divide the human beings into six subgroups: first are the human being who have not yet started using drugs but may start to use drugs if they interact with $I_1$ or $I_2$ known as susceptible and symbolized by $S$, second are the people whose company and environment is risky, these are the people who have started using the drugs but they haven't declared or shown symptoms to be termed as light or heavily addicted users to be referred as exposed and

denoted by $E$. Third and fourth are the people who are light and heavily addicted users who have shown symptoms and capable of influencing another susceptible individual to start using drugs to be referred as Light drug users and heavy drug users and denoted by $I_1$ and $I_2$ respectively, fifth are the the drug users who have started treatment in the rehabilitation centers to be referred to patient and denoted by $Q$ and the last subgroup are the individuals if treated or through self reflection may recover and move to compartment $R$ where by natural practice they are assumed to be temporary invulnerability from drug use before they become susceptible again.

### 2.2. Description of interaction

The close relationship and/or frequency contact of susceptible human and the drug users $I_1$ and $I_2$, increase the chances of becoming a drag user and thus become at high risky subgroup $E$ at a rate $\psi_1$ and $\psi_2$ respectively. Extensive exposure to vulnerable drug use environment lead to light drug user $I_1$ or heavy drug user$I_2$ at a rate $\alpha_1$ and proportional $\rho$ or $(1-\rho)$ respectively. With time the light drug user may become addict to drugs become heavy drug user at a rate $\rho_1\sigma_1$. With self reflection or by changing environment and company light drug addict may recover and become temporally invulnerable to drug use at a rate $\rho_3\sigma_1$, otherwise depending on the nature of the family and the environment they will be taken to the rehabilitation centers for treatment at the rate $\rho_2\sigma_1$. The heavy drug addicts may be taken to the rehabilitation centers for treatment at a rate $\sigma_2$ otherwise if not treated they die due to extensive drug addiction at a rate $\mu_2$. A successful treatment of drug user in the rehabilitation centers lead to recovery at a rate $\alpha_2$, the recovered group become temporally invulnerable to the drug use environment but with time they become susceptible again at a rate $\alpha_3$. Human beings are recruited at a constant rate $\lambda$ and die naturally at the rate $\mu_1$. $\omega$ is the coefficient that represent a control effort using the method of early identification and quarantine of the drug for treatment to reduce the rate of influence that exposes people in the community to drug abuse.

### 2.3. Parameters description

| Parameter | Description |
|---|---|
| $\lambda$ | The rate of recruitment of human population |
| $\psi_1$ & $\psi_2$ | The rate that $S$ become $E$ due to $I_1$ and $I_2$ respectively |
| $\alpha_1$ | Progression rate out of $E$ to drug user states. |
| $\rho$ | Proportional of $E$ becoming $I_2$ |
| $\sigma_1$ | Rate at which individual leave $I_1$ |
| $\rho_1$ | Proportional of $I_1$ becoming $I_2$ |
| $\rho_2$ | Proportional of $I_1$ becoming $Q$ |
| $\rho_3$ | Proportional of $I_1$ becoming $R$ |
| $\alpha_2$ | The rate at which $Q$ become $R$ |
| $\alpha_2$ | The rate at which $R$ become $S$ |
| $\mu_1$ | Natural death rate of human being. |
| $\mu_2$ | Drug addiction induced death rate of human being. |
| $\omega$ | Control efforts to protect $S$ |
| $N$ | Total human population. |

**Table 1:** Parameters description for Drug use model

## 2.4. Model assumption

The drug abuse model is developed by using the assumption listed below:

- Human being in subgroup $S$ are enrolled at a constant rate.
- Same natural death rate for all individuals involved in this study.
- All individual involved in this study have equal chance of becoming drug user.
- On recovery individual become temporary invulnerable to drug abuse.
- All individuals are born susceptible.
- Only heavy drug user may die due to drug addiction.
- Susceptible individual become drug user through social contact process only.

Using the description of interaction and assumption, we summarize the dynamics of drug use in the communities in a compartmental diagram presented in Figure 2.4. The figure captures the interaction between the susceptible, drug users and individuals under treatment.

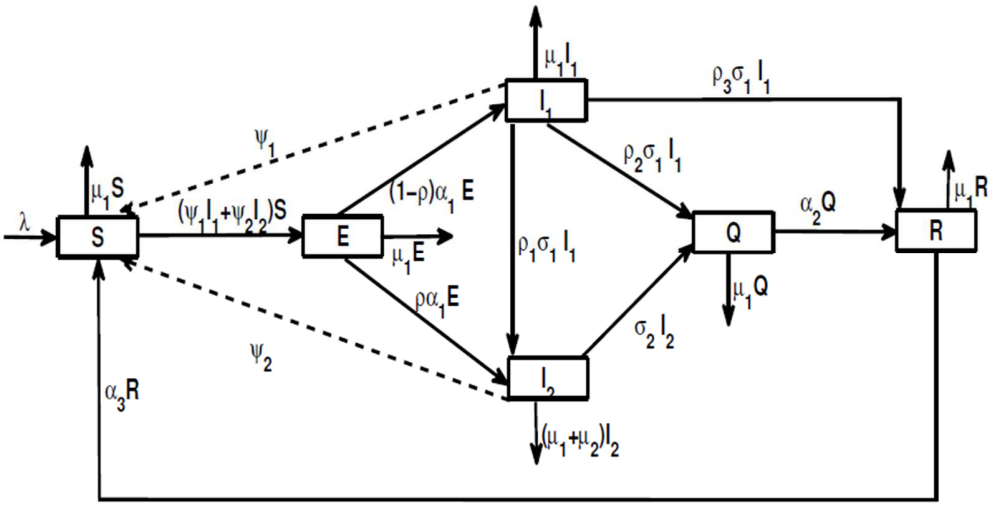


**Figure 1:** Compartmental model for Drug Use

## 2.5. Model equation

The dynamics of drug use in the society is presented by system (2.5)

$$\frac{dS}{dt} = \lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2)(1-\omega)S - \mu_1 S \quad (1)$$

$$\frac{dE}{dt} = (\psi_1 I_1 + \psi_2 I_2)(1-\omega)S - ((1-\rho)+\rho)\alpha_1 E - \mu_1 E \quad (2)$$

$$\frac{dI_1}{dt} = (1-\rho)\alpha_1 E - (\rho_1\sigma_1 + \rho_2\sigma_1 + \rho_3\sigma_1 + \mu_1)I_1 \quad (3)$$

$$\frac{dI_2}{dt} = \rho\alpha_1 E + \rho_1\sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2)I_2 \quad (4)$$

$$\frac{dQ}{dt} = \rho_2\sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1)Q \quad (5)$$

$$\frac{dR}{dt} = \rho_3\sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3)R \quad (6)$$

$$\rho_1 + \rho_2 + \rho_3 = 1$$

## 3. Analysis of the model

Here we analyze the basic drug abuse model (without control $\omega = 0$) is presented. Here the mathematical meaningfulness, existence of stationary point and their stability are established. The basic reproduction number $R_0$ and effective reproduction number $R_e$ are computed using next generation matrix method. We use $R_0$ to define the condition for global and local stability of the stationary points. In the numerical simulation section we depict the dynamical behavior of the drug abuse model with and without control that aim at reducing the effect of drug abuse in the society.

Therefore in this section we consider the drug abuse model when $\omega = 0$ as in the system 3 below;

$$\frac{dS}{dt} = \lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2 + \mu_1)S \quad (7)$$

$$\frac{dE}{dt} = (\psi_1 I_1 + \psi_2 I_2)S - ((1-\rho) + \rho)\alpha_1 E - \mu_1 E \quad (8)$$

$$\frac{dI_1}{dt} = (1-\rho)\alpha_1 E - (\rho_1\sigma_1 + \rho_2\sigma_1 + \rho_3\sigma_1 + \mu_1)I_1 \quad (9)$$

$$\frac{dI_2}{dt} = \rho\alpha_1 E + \rho_1\sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2)I_2 \quad (10)$$

$$\frac{dQ}{dt} = \rho_2\sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1)Q \quad (11)$$

$$\frac{dR}{dt} = \rho_3\sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3)R \quad (12)$$

## 4. Basic properties of the model

### 4.1. Invariant region

Due to the fact that drugs affect human population, then, in order to model this process we need to assume that all of the defined state variables and parameters used in the model are non-negative for $\forall t \geq 0$. We are required to analyze the drug abuse model so that it is defined in a suitable feasible region which has positive state variables and therefore result to Theorem 4.1; The existing forward solutions in $R_+^6$ of the drug abuse model system are feasible $\forall t \geq 0$ whenever they enter the invariant region namely $\Lambda$. where

$$\Lambda = (S, E, I_1, I_2, Q, R) \in R_+^6 : S + E + I_1 + I_2 + Q + R \leq N$$

and $\Lambda$ is the positive invariant region of drug abuse model system

### 4.2. Positivity of the solution

Here we need to show that variables and parameters used in the drug abuse model are non negative $\forall t \geq 0$. Assume the initial values of the drug abuse model system (3) to be: $(S(0) > 0$ and $(E(0), I_1(0), I_2(0), Q(0), R(0)) \geq 0$. Then the model's solution set $S(t), E(t), Q(t), I_1(t), I_2(t)$ and $R(t)$ are positive $\forall t \geq 0$.

## 5. Analysis of the stationary points and $R_0$

This part of the study present the existence of stationary states, the number of drug use cases produced by one drug user in the whole drug user period and stability of the stationary points.

### 5.1. Drug use free equilibrium

We obtain the Drug use free equilibrium by setting $E = Q = I_1 = I_2$ and $R = 0$ and

substitute to the drug abuse model system (3).
The drug use free-equilibrium point of the drug abuse model system is as given in (13)

$$X_0(S^0, E^0, I_1^0, I_2^0, Q^0, R^0) = \left(\frac{\lambda}{\mu_1}, 0,0,0,0,0\right). \quad (13)$$

### 5.2. Basic reproduction number $R_0$ for drug abuse

This is the number of secondary drug abuse cases that are to be produced by one drug user in the whole drug abuse period of that particular individual in a population defined by only susceptible population. The criteria for this dimensionless parameter is that if $R_0 < 1$, then the single drug user in a population defined by only susceptible population may influence less than one individual to start using drugs. This entails that drug abuse problem may be eradicated from the population and the drug free stationary point is asymptotically stable which also means that the drug abuse cannot attack the society.

When $R_0 > 1$ it portray that one drug user in a population defined by only susceptible population may influence more than one individuals to start using drugs. This also means that drug abuse problem may Hence the drug abuse may continue to stay in the society. This situation also means that the drug free equilibrium point is unstable and that it is vividly clear that the drug abuse problem can attack the society and stay for a long time.

And if $R_0 = 1$ it portray that one drug user in entirely susceptible population influence drug use to one new human being. Hence the drug abuse problem will be alive in the society without an serious epidemic as narrated by [1].

To find the basic reproduction number we use next generation method by [19]. Consider a heterogeneous population in compartments $S, E, I_1, I_2, Q$ and $R$ arranged such that $m$ drug using classes come first.

Assume $F_i(x)$ as rate of entrance of new drug users in class $i$, $V_i^+(x)$ rate of transfer of individuals in the class $i$ by any other means except the drug abuse induced $V_i^-(x)$ be the rate of transfer of individuals out of class $i$.

The the model system is as presented below;

$$x_i' = F_i(x) - V_i(x)$$

where $V_i(x) = V_i^-(x) - V_i^+(x)$.

Then we use $x_0$, to find the $m \times m$ matrices $F$ and $V$

$$F = \left(\frac{\partial F_i}{\partial x_j}(x_0)\right), V = \left(\frac{\partial V_i}{\partial x_j}(x_0)\right)$$

with $1 \le i, j \le m$.

By using the study by [5] we call Matrix $FV^{-1}$, a next generation matrix Thus:

$$R_0 = \rho(FV^{-1})$$

$$\frac{dE}{dt} = (\psi_1 I_1 + \psi_2 I_2)S - ((1-\rho) + \rho)\alpha_1 E - \mu_1 E \quad (14)$$

$$\frac{dI_1}{dt} = (1-\rho)\alpha_1 E - (\rho_1\sigma_1 + \rho_1\sigma_1 + \rho_1\sigma_1 + \mu_1)I_1 \quad (15)$$

$$\frac{dI_2}{dt} = \rho E + \rho_1\sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2)I_2 \quad (16)$$

$$\frac{dQ}{dt} = \rho_2\sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1)Q \quad (17)$$

$$\frac{dR}{dt} = \rho_3\sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3)R \quad (18)$$

$$\frac{dS}{dt} = \lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2 + \mu_1)S \tag{19}$$

Now from the system (5.2) the drug using classes are the one with compartment $E, I_1, I_2$ and $Q$ , this will now yield

$$\mathbf{F_i} = \begin{pmatrix} (\psi_1 I_1 + \psi_2 I_2)S \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{20}$$

And

$$\mathbf{V_i} = \begin{pmatrix} \alpha_1 E + \mu_1 E \\ (\rho_1\sigma_1 + \mu_1)I_1 - (1-\rho)\alpha_1 E \\ (\sigma_2 + \mu_1 + \mu_2)I_2 - \rho E - \rho_1\sigma_1 I_1 \\ (\alpha_2 + \mu_1)Q - \rho_2\sigma_1 I_1 - \sigma_2 I_2 \end{pmatrix}. \tag{21}$$

The computation for matrices $F$ and $V$ at $x_0$ are as fiven below;

$$\frac{\partial F_i}{\partial x_j} = \begin{pmatrix} \frac{\partial F_1}{\partial E} & \frac{\partial F_1}{\partial I_1} & \frac{\partial F_1}{\partial I_2} & \frac{\partial F_1}{\partial Q} \\ \frac{\partial F_2}{\partial E} & \frac{\partial F_2}{\partial I_1} & \frac{\partial F_2}{\partial I_2} & \frac{\partial F_2}{\partial Q} \\ \frac{\partial F_3}{\partial E} & \frac{\partial F_3}{\partial I_1} & \frac{\partial F_3}{\partial I_2} & \frac{\partial F_3}{\partial Q} \\ \frac{\partial F_4}{\partial E} & \frac{\partial F_1 4}{\partial I_1} & \frac{\partial F_4}{\partial I_2} & \frac{\partial F_4}{\partial Q} \end{pmatrix} = \begin{pmatrix} 0 & \psi_1 S & \psi_2 S & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now at $x_0$ we will have

$$\mathbf{F} = \begin{pmatrix} 0 & \psi_1 \frac{\lambda}{\mu_1} & \psi_2 \frac{\lambda}{\mu_1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{22}$$

and for $V$ we will have;

$$V = \frac{\partial V_i}{\partial x_j}(x_0)$$

$$\mathbf{V} = \begin{pmatrix} \alpha_1 + \mu_1 & 0 & 0 & 0 \\ -(1-\rho)\alpha_1 & (\rho_1\sigma_1 + \mu_1) & 0 & 0 \\ -\rho\alpha_1 & -\rho_1\sigma_1 & (\sigma_2 + \mu_1 + \mu_2) & 0 \\ 0 & -\rho_2\sigma_1 & -\sigma_2 & (\alpha_2 + \mu_1) \end{pmatrix}. \tag{23}$$

We can obtain $V^{-1}$ and $FV^{-1}$ using maple, then the spectrol radius which is also the basic reproduction number is as in (24) ;

$$R_0 = \frac{\left((\eta\psi_1+\rho\psi_2)\mu_1+\psi_2\rho_1(\eta+\rho)\sigma_1+\eta\psi_1(\sigma_2+\mu_2)\right)\alpha_1\lambda}{\mu_1(\alpha_1+\mu_1)(\sigma_2+\mu_1+\mu_2)(\rho_1\sigma_1+\mu_1)} \quad (24)$$

where $\eta = (1-\rho)$.

When early identification and quarantine control method is applied, the number of secondary cases of drug users will decrease which as a result will reduce the number of adequate contact between the drug user and the susceptible population. Now using the same next generation matrix method we can generate an effective reproduction number that consider the effect of control as in (25)

$$R_e = \frac{\left((\eta\psi_1+\rho\psi_2)\mu_1+\psi_2\rho_1(\eta+\rho)\sigma_1+\eta\psi_1(\sigma_2+\mu_2)\right)(1-\omega)\alpha_1\lambda}{\mu_1(\alpha_1+\mu_1)(\sigma_2+\mu_1+\mu_2)(\rho_1\sigma_1+\mu_1)} \quad (25)$$

where $\eta = (1-\rho)$.

When $\omega \neq 0$ it reduces the value of the number of secondary cases produced as it can be seen in the expression (25), it can be seen that the increase of the control efforts will reduce the value of the expression in (25) which the number of secondary cases of drug users produced by a drug user in the entire period of drug abuse, but when $\omega = 0$ the expression (25) become the basic reproduction number as given in (24).

## 6. Steady state and local stability of the critical points

This section presents the conditions for stability of critical points of the drug abuse model system (3).

### 6.1. Disease free equilibrium

The model has a drug free equilibrium in which we set all infectious compartment and the derivatives equal to zero. Then we have the drug free-equilibrium point is as given in (26)

$$X_0(S^0, E^0, I_1^0, I_2^0, Q^0, R^0) = \left(\frac{\lambda}{\mu_1}, 0,0,0,0,0\right). \quad (26)$$

### 6.2. Local stability of the drug-free equilibrium point

This section presents the analysis for local stability of the drug free stationary point of the drug abuse model. We use Jacobian method by considering that all equations in drug abuse model in (3) are analyzed at the drug free stationary point $X_0$. We are required to compute and asses the eigenvalues of Jacobian matrix in order to verify that the drug free stationary point is locally and asymptotically stable. Further more we need to show that the real parts of the eigenvalues of the matrix at $X_0$ are negative.

Using the concept by [10], we are required to show that eigenvalues are negative, in which we need to prove that determinant of the Jacobian matrix is positive and its trace negative. .

Now using the Jacobian matrix of the system (3) at $X_0$ we can prove that, drug free stationary point $E^0$ is locally asymptotically stable and leads to the following theorem:

$$\mathbf{J(X_0)} = \begin{pmatrix} -\mu_1 & 0 & -\frac{\psi_1\lambda}{\mu_1} & -\frac{\psi_2\lambda}{\mu_1} & 0 & \alpha_3 \\ 0 & -(\alpha_1+\mu_1) & \frac{\lambda\psi_1}{\mu_1} & \frac{\psi_2\lambda}{\mu_1} & 0 & 0 \\ 0 & (1-\rho)\alpha_1 & -(\sigma_1+\mu_1) & 0 & 0 & 0 \\ 0 & \rho\alpha_1 & \rho_1\sigma_1 & -(\sigma_2+\mu_1+\mu_2) & 0 & 0 \\ 0 & 0 & \rho_2\sigma_1 & \sigma_2 & -(\alpha_2+\mu_1) & 0 \\ 0 & 0 & \rho_3\sigma_1 & 0 & \alpha_2 & -(\mu_1+\alpha_3) \end{pmatrix} \quad (27)$$

Using trace and determinant method; we need to check if trace is negative and the determinant of matrix (27) is positive.

Computing we get trace of the matrix (27) as given below;

$$\mathbf{Trace} = -\mu-(\alpha_1+\mu_1)-(\sigma_1+\mu_1)-(\sigma_2+\mu_1+\mu_2)-(\alpha_2+\mu_1)-(\omega+\mu)-(\mu_1+\alpha_3)$$

Therefore the matrix (27) in negative.

We then compute determinant of the matrix (27) in which it is positive if

$$\frac{((\eta\psi_1+\rho\psi_2)\mu_1+\psi_2\rho_1(\eta+\rho)\sigma_1+\eta\psi_1(\sigma_2+\mu_2))\alpha_1\lambda}{\mu_1(\alpha_1+\mu_1)(\sigma_2+\mu_1+\mu_2)(\rho_1\sigma_1+\mu_1)} < 1$$

where

$$\frac{((\eta\psi_1+\rho\psi_2)\mu_1+\psi_2\rho_1(\eta+\rho)\sigma_1+\eta\psi_1(\sigma_2+\mu_2))\alpha_1\lambda}{\mu_1(\alpha_1+\mu_1)(\sigma_2+\mu_1+\mu_2)(\rho_1\sigma_1+\mu_1)} \quad (28)$$

represents the basic reproduction number, $R_0$.

The above results justifies that the drug free stationary point $x^0$ is locally asymptotically stable as in theorem below: The Drug Free stationary point $X_0$ is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

### 6.3. Global stability of the drug-free equilibrium point

This section presents the analysis of the global stability of the drug free stationary point. We use the method by [4] known as Metzler matrix as presented in the steps below;

Assume $\mathbf{Y_n}$ to be the vector of compartments which can not influence any drug user, $\mathbf{Y_i}$ to be the vector for classes that can influence individuals to start using drugs and $\mathbf{Y_{X_0,n}}$ to be the vector of drug free stationary point.

$$\begin{cases} \frac{d\mathbf{Y_n}}{dt} = A_1(\mathbf{Y_n}-\mathbf{Y_{X_0,n}})+A_2\mathbf{Y_i} \\ \frac{d\mathbf{Y_i}}{dt} = A_3\mathbf{Y_i} \end{cases} \quad (29)$$

this will then yield;

$$\mathbf{Y_n} = (S,R,E,Q)^T \quad \mathbf{Y_i} = (I_1,I_2) \quad \mathbf{Y_{X_0,n}} = (\frac{\vartheta}{\mu},0)$$

$$\mathbf{Y_n} - \mathbf{Y_{X_{0,n}}} = \begin{pmatrix} S - \frac{\lambda}{\mu_1} \\ R \\ E \\ Q \end{pmatrix}$$

Using the Metziler matrix method we are required to show that the eigen values of a matrix $A_1$ are negative and $A_3$ is a Metzler matrix which represent a matrix with non-negative off diagonal element. The subsystem (29) then becomes;

$$\begin{pmatrix} \lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2 + \mu_1)S, \\ \rho_3\sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3)R. \\ (\psi_1 I_1 + \psi_2 I_2)S - ((1-\rho) + \rho)\alpha_1 E - \mu_1 E. \\ \rho_2\sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1)Q. \end{pmatrix} = A_1 \begin{pmatrix} S - \frac{\vartheta}{\mu} \\ R \\ E \\ Q \end{pmatrix} + A_2 \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

and

$$\begin{pmatrix} (1-\rho)\alpha_1 E - (\rho_1\sigma_1 + \rho_2\sigma_1 + \rho_3\sigma_1 + \mu_1)I_1, \\ \rho\alpha_1 E + \rho_1\sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2)I_2. \end{pmatrix} = A_3 \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

We then use the drug use influencing and non-influencing element from the general Drug abuse model model to get matrices (30):

$$\mathbf{A_1} = \begin{pmatrix} -\mu_1 & \alpha_3 & 0 & 0 \\ 0 & -(\mu_1 + \alpha_3) & 0 & \alpha_2 \\ 0 & 0 & -(\alpha_1 + \mu_1) & 0 \\ 0 & 0 & 0 & -(\alpha_2 + \mu_1) \end{pmatrix} \quad (30)$$

$$\mathbf{A_2} = \begin{pmatrix} -\frac{\lambda\psi_1}{\mu_1} & -\frac{\lambda\psi_2}{\mu_1} \\ \rho_3\sigma_1 & 0 \\ \frac{\lambda\psi_1}{\mu_1} & \frac{\lambda\psi_2}{\mu_1} \\ \rho_2\sigma_1 & \sigma_2 \end{pmatrix} \quad (31)$$

$$\mathbf{A_3} = \begin{pmatrix} -(\sigma_1 + \mu_1) & 0 \\ \rho_1\sigma_1 & -(\sigma_2 + \mu_1 + \mu_2) \end{pmatrix} \quad (32)$$

It is clearly seen that eigenvalues values of a matrix $A_1$, are real and negative.This justifies that the system

$$\frac{d\mathbf{Y_n}}{dt} = A_1(\mathbf{Y_n} - \mathbf{Y_{X_{0,n}}}) + A_2\mathbf{Y_i}$$

is globally and asymptotically stable at $\mathbf{Y_{X_0}}$.

We can also see that; $A_3$ has non-negative off-diagonal elements which means it is a Metzler stable matrix. Therefore Drug Free Equilibrium point for drug abuse model system is globally asymptotically stable as in the theorem below: The drug-free equilibrium point is globally asymptotically stable in $E_0$ if $R_0 < 1$ and unstable if $R_0 > 1$.

### 6.4. Existence of drug abuse endemic equilibrium

In this section we establish the conditions for existence of the drug abuse endemic equilibrium point of the system (3).

To obtain the drug abuse endemic equilibrium point $E^*(S^*, E^*, I_1^*, I_2^*, Q^*, R^*)$ we set system (3) equal to zero and solve equations.

The model system (6.4) which exist for $R_O > 1$.

$$\lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2 + \mu_1)S = 0 \quad (33)$$
$$(\psi_1 I_1 + \psi_2 I_2)S - ((1-\rho) + \rho)\alpha_1 E - \mu_1 E = 0 \quad (34)$$
$$(1-\rho)\alpha_1 E - (\rho_1\sigma_1 + \rho_2\sigma_1 + \rho_3\sigma_1 + \mu_1)I_1 = 0 \quad (35)$$
$$\rho\alpha_1 E + \rho_1\sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2)I_2 = 0 \quad (36)$$
$$\rho_2\sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1)Q = 0 \quad (37)$$
$$\rho_3\sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3)R = 0 \quad (38)$$

Studies by [18] and that of [11] portray how to we prove the existence the drug abuse endemic equilibrium points in which we are required to satisfy the condition $E \neq 0$ or $Q \neq 0$ or $I_1 \neq 0$ or $I_2 \neq 0$ that is $S > 0$ or $E > 0$ or $Q > 0$ or $I_1 > 0$ or $I_2 > 0$ or $R > 0$. Adding the equation in the system (6.4) yields (39)

$$\lambda - \mu_1(S + E + I_1 + I_2 + Q + R) - \mu_2 I_2 = 0 \quad (39)$$

We then have

$$\lambda = \mu_1 N + \mu_2 I_2$$

Therefore since $\lambda > 0$, $\mu_1 > 0$ and $\mu_2 > 0$ we can discern that $\mu_1 N > 0$ and $\mu_2 I_2 > 0$ implying that $S > 0$, $E > 0$, $I_1 > 0$, $I_2 > 0$, $Q > 0$ and $R > 0$ .

This prove that the drug abuse endemic equilibrium point of the drug abuse model system exists.

### 6.5. Global stability of drug abuse endemic equilibrium point

This section analyzes the condition for stability of the drug abuse endemic equilibrium points. The results in the study by [19], shows that the local stability of the Drug Free Equilibrium implies that the local stability of the Drug Abuse Endemic Equilibrium is also locally stable for the reverse condition.

We use Korobeinikov approach as described by [?, ?, ?] to find the global stability of Drug Abuse Endemic stationary point.

We thus formulate a suitable Lyapunov function for drug abuse model system in the following form:

$$V = \sum a_i(y_i - y_i^* \ln y_i)$$

where $a_i$ is an appropriately selected positive constant, $y_i$ is a population of the $i^{th}$ class type, and $y_i^*$ is the stationary point.

We will then have

$$V = W_1(S - S^*\ln S) + W_2(E - E^*\ln E) + W_4(I_2 - I^*1s\ln I_1) + W_5(I_2 - I_a^*\ln I_2) + W_3(Q - Q^*\ln Q) + W_6(R - R^*\ln R)$$

The constants $W_i$ are non-negative in $\Lambda$ such that $W_i > 0$ for $i = 1,2,3\ldots6$. $V$ which is the Lyapunov function $W_1, W_2, \ldots, W_6$ defined as the Lyapunov function constant are selected so that $V$ should be continuous and differentiable in a space.

Finding the time derivative of $V$ we get:

$$\frac{dV}{dt} = W_1(1-\frac{S^*}{S})\frac{dS}{dt} + W_2(1-\frac{E^*}{E})\frac{dE}{dt} + W_3(1-\frac{I_1^*}{I_1})\frac{dI_1}{dt} + W_4(1-\frac{I_2^*}{I_2})\frac{dI_2}{dt} + W_5(1-\frac{Q^*}{Q})\frac{dQ}{dt} + W_6(1-\frac{R^*}{R})\frac{dR}{dt}$$

Substituting the respective equations from the drug abuse model system (3) we get;

$$\begin{aligned}\frac{dV}{dt} = {} & W_1(1-\frac{S^*}{S})[\lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2 + \mu_1)S] \\ & + W_2(1-\frac{E^*}{E})[(\psi_1 I_1 + \psi_2 I_2)S - ((1-\rho)+\rho)\alpha_1 E - \mu_1 E] \\ & + W_3(1-\frac{I_1^*}{I_1})[(1-\rho)\alpha_1 E - (\rho_1\sigma_1 + \rho_2\sigma_1 + \rho_3\sigma_1 + \mu_1)I_1] \\ & + W_4(1-\frac{I_2^*}{I_2})[\rho\alpha_1 E + \rho_1\sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2)I_2] \\ & + W_5(1-\frac{Q^*}{Q})[\rho_2\sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1)Q] \\ & + W_6(1-\frac{R^*}{R})[\rho_3\sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3)R]\end{aligned}$$

Computing time derivative of $V$ at drug abuse endemic stationary point yields:

$$\begin{aligned}\frac{dV}{dt} = {} & -W_1(1-\frac{S^*}{S})^2 - W_2(1-\frac{E^*}{E})^2 - W_3(1-\frac{I_1^*}{I_1})^2 \\ & - W_4(1-\frac{I_2^*}{I_2})^2 - W_5(1-\frac{Q^*}{Q})^2 - W_6(1-\frac{R^*}{R})^2 \\ & + F(S,E,I_1,I_2,Q,R)\end{aligned}$$

where the function $F(S,E,I_1,I_2,Q,R)$ is non positive,

Using the insight from the study by [14] and [8]. We have $F(S,E,I_1,I_2,Q,R) \leq 0$ for all $S,E,I_1,I_2,Q,R$, Then $\frac{dV}{dt} \leq 0$ for all $S,E,I_1,I_2,Q,R$ and it is zero when $S = S^*, E = E^*, Q = Q^*, I_1 = I_1^*, I_2 = I_2^*, R = R^*$ Hence the largest compact invariant set in $S,E,I_1,I_2,Q,R$ such that $\frac{dV}{dt} = 0$ is the singleton $E^*$ which is Drug Abuse Endemic stationary point of the model system (3).

By LaSalles's invariant principle in [9] we proclaim that $E^*$ is globally asymptotically stable in the interior of drug abuse model system region of $S,E,I_1,I_2,Q,R$ as in Theorem 6.5

If $R_0 > 1$ then the drug abuse model system (3) has a distinctive drug abuse endemic equilibrium point $E^*$ which is globally asymptotically stable in $S,E,I_1,I_2,Q,R$

## 7. Numerical analysis and simulation

The section below presents parameter values, numerical analysis and simulation of the drug abuse model, it shows the dynamical behaviour of the drug abuse in the society over the particular period of time.

### 7.1. Parameter values

The values of the parameters used for numerical simulation in the drug abuse model are presented in Table 2. The parameters that are obtained from existing related articles and reports and some are estimated.

| Parameters | Value | Source |
|---|---|---|
| $\lambda$ | 5 | Estimated |
| $\psi_1$ | 0.0001 | Estimated |
| $\psi_2$ | 0.0003 | Estimated |
| $\alpha_1$ | 0.039 | [6] |
| $\rho$ | 0.2 | Estimated |
| $\sigma_1$ | 0.0301 | Estimated |
| $\rho_1$ | 0.4 | Estimated |
| $\mu_1$ | 0.02 | [3] |
| $\rho_2$ | 0.3 | .Estimated |
| $\rho_3$ | 0.3 | Estimated |
| $\alpha_2$ | 0.0051 | Estimated |
| $\alpha_3$ | 0.0089 | Estimated |
| $\mu_2$ | 0.059 | [16] |

**Table 2:** Parameters values for Drug Abuse model.

### 7.2. Results and discussion

Figure 7.2 shows the dynamics of human population when there is drug abuse in the community. We depict the dynamical behavior of the population when no control effort is applied to reduce or eliminate the problem. The simulation shows the significant increase of the drug users both light and heavy at a very high rate. The massive growth of drug users proportionally decrease the susceptible individuals as most of them will be exposed to the environment that may influence drug abuse through adequate contact with the drug users. As assumed adequate and/or frequent contact between the susceptible and drug user may influence the susceptible to start using drug. That is to say the increase of the drug users will increase the rate of contact between the class $S$ and the drug users and thus increase the number of drug users. This is justified by the result in Figure 7.2 in which due to the increase of the rate of drug influence the number of human exposed to disease also increase at a very high rate.

Although the control is not taken seriously as there are no serious effort applied to control the drug abuse apart from the establishment of rehabilitation centers. When the number of drug user increase the number of individual who will be taken to the rehabilitation centers and eventually recover will also increase significantly. Figure 7.2 show the increase of the number of people under treatment and those who recover through changing mind or after treatment. As the problem gets serious through natural mechanisms society will eventually find ways to rescue ad hence justify the increase of individual in class $Q$ and $R$.

Depending on the environment and the individual's resistance to drug addiction, after a certain period of time the light drug user become heavy drug user. This occurs when a light drug user is not taken for treatment in the incubation centers timely. This is to say when there is no efficiency control mechanism the number of heavy drug user increases significantly. Figure 7.2 also shows the high rate of increase of heavy drug user over time as most of the light drug users will graduate and become heavy drug users due to delayed treatment in the incubation centers.

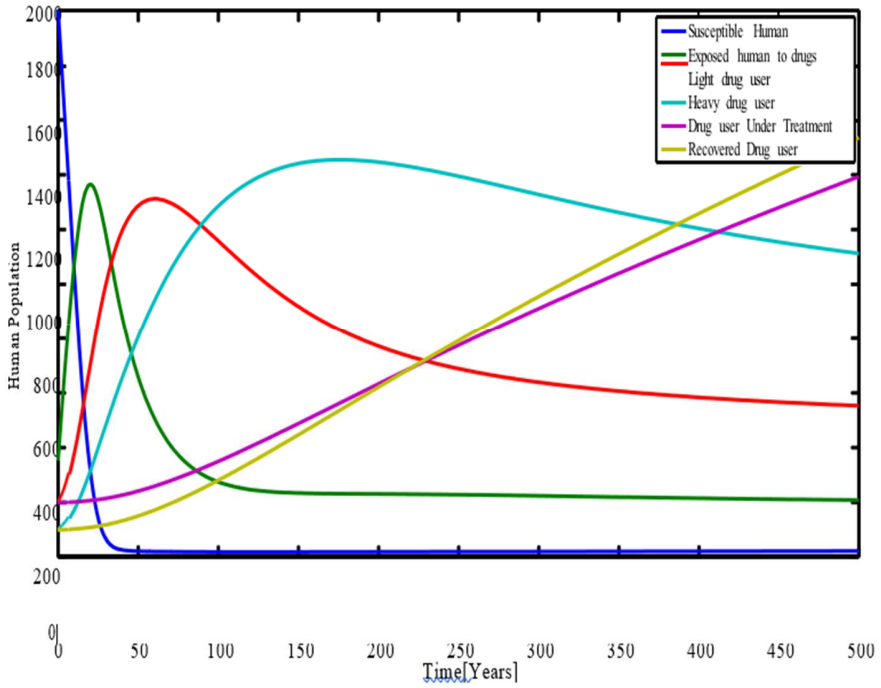


**Figure 2:** Behavioral Dynamics of Drug Abuse in Human Population

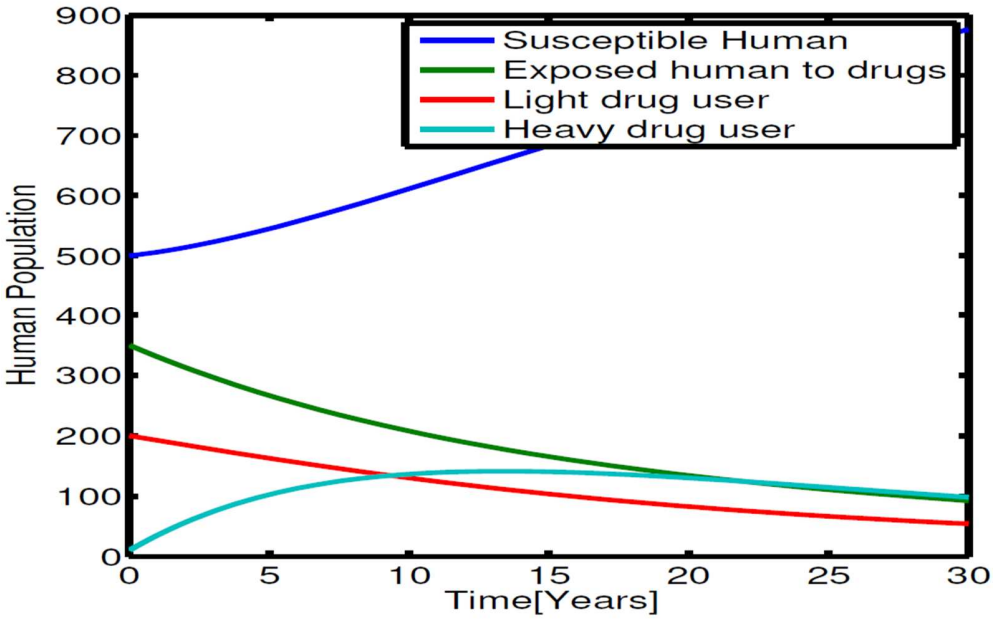


**Figure 3:** Effect of control Method

The increase of drug abuse in many society is mostly due to delayed treatment that allow the interaction between the drug user and the susceptible individual. The delayed treatment also lead to an increase of heavy drug users that as a result increase the number of death due to drug abuse. This alert the need to apply control mechanism that would reduce the effect of delayed treatment. As a solution to the problem we apply the the early identification and quarantine method as a control method that identifies the drug user earlier and quarantine them for treatment to reduce their influence to the society and protect them from becoming heavy drug addict (see drug abuse model system 2.5). Figure 7.2 shows the significant increase of the susceptible human being and a substantial decrease of light and heavy drug users. This dynamical behavior is due to the application of control method that prevent the drug use behavior to affect the susceptible population.

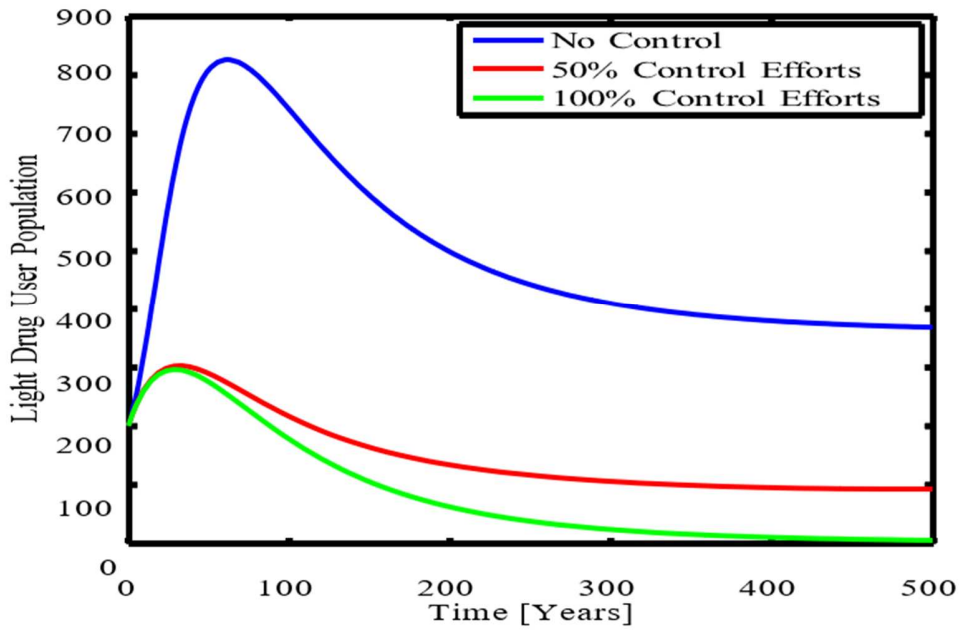


**Figure 4:** Effect of Variation of level of control in Light Drug Users

The result of any control method depend on the level of effort invested in it, when the effort is high the control results also become higher otherwise the control results will be low. The level of the control method is measured by the efficiency of the control method, the effective point and time the control is applied and the length of the control period. Figure 7.2 and Figure 7.2 shows the effect of different level of effort applied to control drug abuse in the community.

The result in the two figures shows the significant reduction of the drug users when the control effort is high, but it also gets weaker when the control efforts is reduced. In a case of drug abuse, the positive control results is the result of the timely application of the control method that hinder secondary cases from happening. It deter the rate at which the drug user population influence the drug using behavior to the susceptible population.

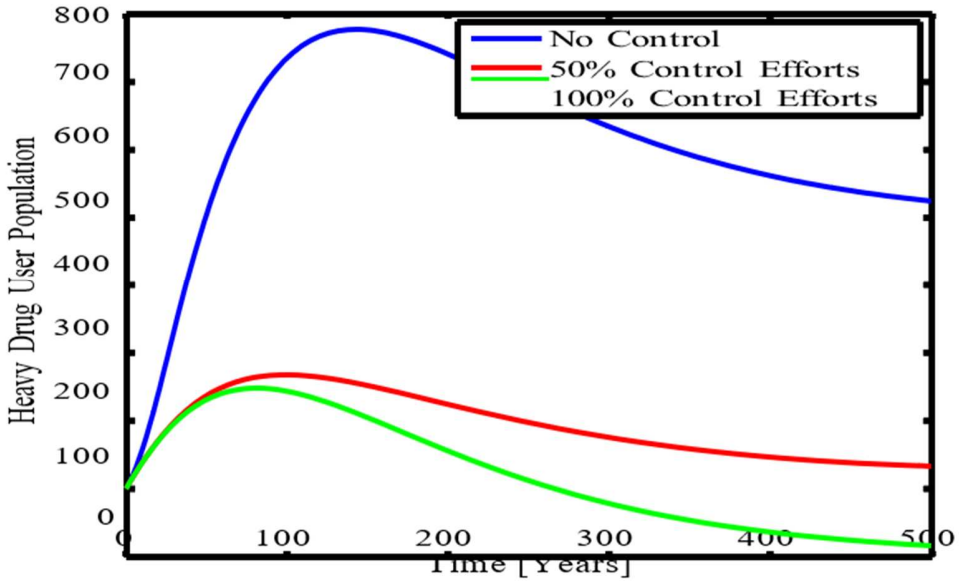


**Figure 5:** Effect of Variation of level of control in Heavy Drug Users

## 8. Conclusion

In this study, the mathematical model to study the dynamical behavior of the community with drug abuse is developed and analyzed. We formulate and analyze the model by establishing the basic and significant condition for the model. The analytical result shows that the drug abuse model is mathematically meaningful and defined in the positive region $\Lambda$. We establish conditions for existence of the equilibrium state and found both drug free and drug use endemic equilibrium point to be locally and asymptotically stable when $R_0 < 1$ and unstable otherwise. Numerical simulation results shows the increase in the drug user (both light and heavy) and the exposed population when the there is no significant control applied to reduce the problem. The simulation further the significant decrease of the rate at which new drug users are influenced and thus reduce number of drug users in the community which then increase the number of susceptible human who in a way are not risky of becoming drug users.

Numerical result further indicate that, the effective result of the control method depends on the effort applied. When the effort is high the chance to defeat the problem gets higher. The control method reduces the progression rate of the light drug users to heavy drug uses. It is beyond doubt that the involvement of society is very vital to the successful control of drug abuse in society. This will help to get a holistic understanding of the nature and practice of the drug users and most people involved in the drug abuse business.

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**Authors' Contributions.** All the authors contributed equally to this work.

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