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Approximative Analysis for Boundary Value Problems of Fractional Order via Topological Degree Method

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Abstract. This paper investigates the existence and uniqueness of a solution to the boundary value problems involving the Caputo fractional derivative in Banach space. We are applying the topological degree approach and fixed point theorem with topological structures in some appropriate situations. An example is propounded to uphold our outcomes.

Keywords**:** Fractional derivatives and integrals; Topological properties of mappings and Fixed point theorems.

AMS Mathematics Subject Classification (2010): 26A33; 47H10; 58K15

1. Introduction

Fractional differential equations are established to be effective modeling of the many phenomena in several fields of science for more details, see [14, 15, 17, 18, 26]. Certainly, the usage of topological strategies stands up very close to evaluating the existence of solutions for fractional differential equations within the last decades, see [12], and [16]. Fractional differential equations in Banach space are receiving more attention by means of many researchers such as Agarwal et al. [3, 4], Balachandran and Park [6], Benchohra et al. [7] and Zhang [24]. Boundary value problems with integral boundary conditions establish a very significant class of problems. They include two, three, multipoint and nonlocal boundary value problems as special cases [10, 11]. Integral boundary conditions appear in cellular systems [2] and population dynamics [8]. In 2006, Zhang [25], considered the existence of positive solutions for nonlinear fractional boundary value problems via applying the properties of the green function and fixed point theorem on cones. In 2009, Benchohra et al. [7], examined the existence and uniqueness of solutions for fractional boundary value problems with nonlocal conditions by fixed point theorem. In 2012, Wang et. al [22, 23], obtained the required and sufficient conditions for fractional boundary value problems via a coincidence degree for condensing maps in Banach spaces. In 2015, the result was extended to the case of solutions to the fractional-order multipoint boundary value problem by Khan and Shah [13], who intentioned sufficient conditions for existence results for the boundary value problem. In 2017, Samina et al. [20], studied the existence of solutions of nonlinear fractional Hybrid differential equations by some results on the existence of solutions and therefore Kuratowski's measure of non-compactness. Abdol and Panchal [1], proved a new uniqueness of results for nonlinear integro-differential equations with respect to the Caputo fractional operator fractional. They used fractional calculus and its properties, Banach contraction mapping principle and Bihari's inequality. Li and Zhai [19], investigated the existence and uniqueness of solutions for Langevin equations with two fractional orders by using e -positive operators and Altman fixed point theory. In [9], the authors considered the initial problem for systems of differential equations to fractional order. They produced a regularization problem and were given an algorithm for normal and unique solubility general iterative systems of differential equations with partial derivatives.

Motivated from the above-cited results, our aim during this paper is to verify some new outcomes on the following boundary value problem (BVP) for fractional differential equations involving the Caputo fractional derivative by topological degree method and fixed point theorem in Banach space $\mathcal{X}$.

$$\begin{cases} {}^{c}D^{q}x(t) = \xi(t, x(t)) \\ x(0) = \eta(x), \quad x(T) = x_0 \end{cases} \tag{1}$$

where $t \in \mathcal{J} := [0, T]$, $q \in (0,1)$, ${}^{c}\mathcal{D}^{q}$ is the Caputo derivative, $\xi: \mathcal{J} \times \mathcal{X} \to \mathcal{X}$ and $\eta: \mathcal{X} \to \mathcal{X}$ are given continuous maps.

2. Preliminaries

In this section, we introduce some necessary definitions, propositions and theorems which are needed throughout this paper.

We define a Banach space $\mathcal{C}(\mathcal{J}, \mathcal{X})$ as a Banach space of all continuous functions from $\mathcal{J}$ into $\mathcal{X}$ with the topological norm $\| x \|_{c} := sup\{\| x(t) \| : x \in \mathcal{C}(\mathcal{J}, \mathcal{X}),\ t \in \mathcal{J}\}$ and $\mathcal{J} = [0, T], T > 0$.

Definition 2.1. *([17], [21]) For a given function ξ on the closed interval $[a, b]$, the qth fractional order integral of ξ is defined by;*

$$\mathcal{I}_{a+}^{q}\xi(t) \quad = \frac{1}{\Gamma(q)}\int_{a}^{t}(t-s)^{q-1}\xi(s)ds, \tag{2}$$

where Γ is the gamma function.

Definition 2.2. *([17], [21]) For a given function ξ on the closed interval $[a, b]$, the qth Riemann- Liouville fractional-order derivative of ξ, is defined by;*

$$(\mathcal{D}_{a+}^{q}\xi)(t) \quad = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}(t-s)^{n-q-1}\xi(s)ds. \tag{3}$$

where $n = [q] + 1$ and $[q]$ denotes the integer part of q.

Definition 2.3. *([17], [21]) For a given function ξ on the closed interval $[a, b]$, the Caputo fractional order derivative of ξ, is defined by;*

$$({}^{c}\mathcal{D}_{a+}^{q}\xi)(t) \quad = \frac{1}{\Gamma(n-q)}\int_{a}^{t}(t-s)^{n-q-1}\xi^{(n)}(s)ds, \tag{4}$$

where $n = [q] + 1$.

Definition 2.4. *([21], [26]) Let $\Omega \subset X$ and $F: \Omega \to X$ be a continuous bounded map. One can say that F is α-Lipschitz if there exists $k \geq 0$ such that*

$$\alpha(F(B)) \leq k\alpha(B) \quad (\forall) \ B \subset \Omega \ \ bounded\,.$$

In case, $k < 1$, then we call F is a strict α-contraction. One can say that F is α-condensing if

$$\alpha(F(B)) < \alpha(B) \quad (\forall) \ B \subset \Omega \ \ bounded \ with \ \ \alpha(B) > 0.$$

We recall that $F: \Omega \to X$ is Lipschitz if there exists $k > 0$ such that

$$\| F_x - F_y \| \leq k \| x - y \| \quad (\forall) \ x, y \subset \Omega,$$

and if $k < 1$ then F is a strict contraction.

Theorem 2.1. *([21]) Let $\mathcal{X}$ be a Banach space, and $\mathcal{F}, \mathcal{G}: \mathcal{X} \to \mathcal{X}$ be two operators such that $\mathcal{F}$ is a contraction operator, and $\mathcal{G}$ is a completely continuous operator then the operator equation $\mathcal{T}x = \mathcal{F}x + \mathcal{G}x = x$ has a solution $x \in \mathcal{X}$.*

Proposition 2.1. *([21], [26]) If $\mathcal{F}, \mathcal{G}: \Omega \to X$ are α-Lipschitz maps with constants k, k' respectively, then $\mathcal{F} + \mathcal{G}: \Omega \to X$ is α-Lipschitz with constant $k + k'$.*

Proposition 2.2. *([21], [26]) If $\mathcal{F}: \Omega \to X$ is compac, then $\mathcal{F}$ is α-Lipschitz with zero constant.*

Proposition 2.3 *([21], [26])) If $\mathcal{F}: \Omega \to X$ is Lipschitz with constant k, then $\mathcal{F}$ is α-Lipschitz with the same constant k.*

3. Existence and uniqueness result of the system

First, we define the meaning of a solution to the BVP(1).

Definition 3.1. *A function $x \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ is called a solution of the fractional BVP(1), if x satisfies the equation ${}^c\mathcal{D}^q x(t) = \xi(t, x(t))$ almost everywhere on $\mathcal{J}$ and the conditions $x(0) = \eta(x)$, $x(T) = x_0$.*

In order to solve a problem of existence to BVP (1), we need the following assumptions:

[H1] $\xi: \mathcal{J} \times \mathcal{X} \to \mathcal{X}$ is continuous.

[H2] There exists a constant $\delta_\xi \in (0,1)$, such that

$$\| \xi(t, x) - \xi(t, y) \| \leq \delta_\xi \| x - y \|, \quad \forall t \in \mathcal{J}, x, y \in \mathcal{X}.$$

[H3] There exists a constant $\delta_\eta \in (0,1)$, such that

$$\| \eta(x) - \eta(y) \| \leq \delta_\eta \| x - y \|, \quad \forall x, y \in \mathcal{X}.$$

Lemma 3.1. *Let $0 < q < 1$, the fractional integral equation*

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \xi(s, x(s)) ds - \frac{t}{T\Gamma(q)} \int_0^T (T-s)^{q-1} \xi(s, x(s)) ds \\ -(\frac{t}{T} - 1)\eta(x) + \frac{t}{T} x_0 \tag{5}$$

has a solution $x \in \mathcal{C}(\mathcal{J},\mathcal{X})$ if and only if x is a solution of the fractional BVP (1).
Proof: Assume that x is a solution of BVP(1), then we have to show that x is also a solution of FIE(5). We have,

$$x(t)-x(0)=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\xi(s,x(s))ds \tag{6}$$

Then,

$$x(T)-x(0)=\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}\xi(s,x(s))ds$$

$$tx(T)-tx(0)=\frac{t}{\Gamma(q)}\int_0^T (T-s)^{q-1}\xi(s,x(s))ds$$

$$Tx(0)+tx(T)-tx(0)=Tx(0)+\frac{t}{\Gamma(q)}\int_0^T (T-s)^{q-1}\xi(s,x(s))ds$$

By the boundary conditions $x(0)=\eta(x),\ x(T)=x_0$, we get

$$x(0)=-\frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1}\xi(s,x(s))ds-(\frac{t}{T}-1)\eta(x)+\frac{t}{T}x_0 \tag{7}$$

Replacing in equation(6), we get

$$x(t)=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\xi(s,x(s))ds-\frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1}\xi(s,x(s))ds-(\frac{t}{T}-1)\eta(x)+\frac{t}{T}x_0$$

Conversely, assume that $x \in \mathcal{C}(\mathcal{J},\mathcal{X})$ satisfies the FIE(5). If $t=0$, it is easy to obtain $x(0)=\eta(x),\ x(T)=x_0$. For $t \in J$ by using the both facts that ${}^cD^q$ is the left inverse of I_t^q and ${}^cD^q$ of a constant is equal to zero then we get ${}^c\mathcal{D}^q x(t)=\xi(t,x(t))$ which completes the proof.

Theorem 3.1. *Assume that (H1)-(H3) hold, if*

$$\frac{2\delta_\xi T^q}{\Gamma(q+1)}+\delta_\eta<1.$$

then the fractional BVP (1) has a unique solution $x \in \mathcal{C}(\mathcal{J},\mathcal{X})$.
Proof: First, Consider the operator $\mathcal{F}:\mathcal{C}(\mathcal{J},\mathcal{X}) \to \mathcal{C}(\mathcal{J},\mathcal{X})$ defined by

$$\mathcal{F}(x)(t)=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\xi(s,x(s))ds-\frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1}\xi(s,x(s))ds-(\frac{t}{T}-1)\eta(x)+\frac{t}{T}x_0.$$

The fixed points of the operator $\mathcal{F}$ are solutions of the problem BVP(1). Then,

$$\begin{aligned}\|\mathcal{F}(x)(t)-\mathcal{F}(y)(t)\| &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|\xi(s,x(s))-\xi(s,y(s))\|ds\\ &+\frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1}\|\xi(s,x(s))-\xi(s,y(s))\|ds+(\frac{t}{T}-1)\|\eta(x)-\eta(y)\|\\ &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\delta_\xi\|x-y\|+\frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1}\delta_\xi\|x-y\|ds\\ &+(\frac{t}{T}-1)\delta_\eta\|x-y\|\\ &\leq \frac{1}{\Gamma(q)}(\frac{t^q}{q})\delta_\xi\|x-y\|+\frac{t}{T\Gamma(q)}(\frac{T^q}{q})\delta_\xi\|x-y\|+(\frac{t}{T}-1)\delta_\eta\|x-y\|\end{aligned}$$

$$\leq [\frac{\delta_\xi t(t^{q-1}+T^{q-1})}{\Gamma(q+1)} + \delta_\eta(\frac{t}{T}-1)] \parallel x-y \parallel \tag{8}$$

as $0 \leq t \leq T$ then,

$$\parallel \mathcal{F}(x)(t) - \mathcal{F}(y)(t) \parallel \leq [\frac{2\delta_\xi T^q}{\Gamma(q+1)} + \delta_\eta] \parallel x-y \parallel$$

Hence, $\mathcal{F}$ is a contraction mapping on $\mathcal{C}(\mathcal{J},\mathcal{X})$ with contraction constant $[\frac{2\delta_\xi T^q}{\Gamma(q+1)} + \delta_\eta]$. By applying Banach's contraction mapping principle, we deduce that the operator $\mathcal{F}$ has a unique fixed point on $\mathcal{C}(\mathcal{J},\mathcal{X})$ which implies the BVP(1) has a unique solution on $\mathcal{C}(\mathcal{J},\mathcal{X})$.

Theorem 3.2. *Assume that* $(H1)-(H3)$ *and the following hypotheses:*

[H4] There exist $\delta_1, \delta_2 > 0$ and $q_1 \in [0,1)$ such that

$$\parallel \xi(t,x) \parallel \leq \delta_1 \parallel x \parallel^{q_1} + \delta_2, \quad \forall (t,x) \in \mathcal{J} \times \mathcal{X}.$$

[H5] There exist $\delta_3, \delta_4 > 0$, $q_2 \in [0,1)$ such that

$$\parallel \eta(x) \parallel \leq \delta_3 \parallel x \parallel^{q_2} + \delta_4, \quad \forall x \in \mathcal{X}.$$

hold then the fractional BVP(1) has at least one solution $x \in \mathcal{C}(\mathcal{J},\mathcal{X})$.
Proof: The proof will be presented in the following steps:

Step 1: Prove continuity of $\mathcal{F}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of a bounded set $\mathcal{B}_\iota \subseteq \mathcal{C}(\mathcal{J},\mathcal{X})$ such that $\parallel x_n - x \parallel \to 0$ as $n \to \infty$ in $\mathcal{B}_\iota(\iota > 0)$. For all $s \in [0,t]$, $t \in \mathcal{J}$, we have to show that $\parallel \mathcal{F}x_n - \mathcal{F}x \parallel \to 0$ as $n \to \infty$ as follows:

$$\parallel (\mathcal{F}x_n)(t) - (\mathcal{F}x)(t) \parallel \leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} \parallel \xi(s,x_n(s)) - \xi(s,x(s)) \parallel ds$$
$$+\frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1} \parallel \xi(s,x_n(s)) - \xi(s,x(s)) \parallel ds + (\frac{t}{T}-1) \parallel \eta(x_n) - \eta(x) \parallel$$
$$\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\delta_\xi \parallel x_n - x \parallel ds + \frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1}\delta_\xi \parallel x_n - x \parallel ds$$
$$+(\frac{t}{T}-1)\delta_\eta \parallel x_n - x \parallel \to 0 \ \ as \ \ n \to \infty.$$

Step 2: Prove $\mathcal{F}$ map bounded sets into bounded sets in $\mathcal{C}(\mathcal{J},\mathcal{X})$.
For any $\iota > 0$, we have $x \in \mathcal{B}_\iota := \{x \in \mathcal{C}(\mathcal{J},\mathcal{X}) : \parallel x \parallel \leq \iota\}$,

$$\parallel (\mathcal{F}x)(t) \parallel \leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} \parallel \xi(s,x(s)) \parallel ds + \frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1} \parallel \xi(s,x(s)) \parallel ds$$
$$+(\frac{t}{T}-1) \parallel \eta(x) \parallel + \frac{t}{T}x_0$$
$$\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[\delta_1 \parallel x \parallel^{q_1} + \delta_2]ds + \frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1}[\delta_1 \parallel x \parallel^{q_1} + \delta_2]ds$$
$$+\left(\frac{t}{T}-1\right)[\delta_3 \parallel x \parallel^{q_2} + \delta_4] + \frac{t}{T}x_0$$

$$\leq \frac{1}{\Gamma(q)[\delta_1 \parallel x \parallel^{q_1} + \delta_2]\left(\frac{t^q}{q}\right)} + \frac{t}{T\Gamma(q)[\delta_1 \parallel x \parallel^{q_1} + \delta_2]\left(\frac{T^q}{q}\right)}$$
$$+\left(\frac{t}{T} - 1\right)[\delta_3 \parallel x \parallel^{q_2} + \delta_4] + \frac{t}{T}x_0$$
$$\leq \frac{t(t^{q-1} + T^{q-1})}{\Gamma(q+1)[\delta_1 \parallel \iota \parallel^{q_1} + \delta_2]} + \left(\frac{t}{T} - 1\right)[\delta_3 \parallel \iota \parallel^{q_2} + \delta_4] + \frac{t}{T}x_0 \coloneqq k$$

Thus, $\mathcal{F}$ map bounded sets into bounded sets in $\mathcal{C}(\mathcal{J},\mathcal{X})$.

Step 3*:* Prove $\mathcal{F}(\mathcal{B}_\iota)$ is equicontinuous. For $t_1, t_2 \in \mathcal{J}$ and $0 \leq t_1 \leq t_2 \leq 1$, let $x \in \mathcal{B}_\iota$, then,

$$\parallel (\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1) \parallel = \parallel \frac{1}{\Gamma(q)}\int_0^{t_2}(t_2 - s)^{q-1}\xi(s,x(s))ds - \frac{t_2}{T\Gamma(q)}\int_0^T (T-s)^{q-1}\xi(s,x(s))ds$$
$$-(\frac{t_2}{T})\eta(x) + \frac{t_2}{T}x_0 - \frac{1}{\Gamma(q)}\int_0^{t_1}(t_1 - s)^{q-1}\xi(s,x(s))ds + \frac{t_1}{T\Gamma(q)}\int_0^T (T-s)^{q-1}\xi(s,x(s))ds$$
$$+(\frac{t_1}{T})\eta(x) - \frac{t_1}{T}x_0 \parallel$$
$$\leq \frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] \parallel \xi(s,x(s)) \parallel ds + \frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2 - s)^{q-1}\xi(s,x(s))ds$$
$$+\frac{(t_2 - t_1)}{T\Gamma(q)}\int_0^T (T-s)^{q-1} \parallel \xi(s,x(s)) \parallel ds + \frac{(t_2 - t_1)}{T} \parallel \eta(x) \parallel + \frac{(t_2 - t_1)}{T}x_0$$

As $t_1 \to t_2$, we get $\parallel (\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1) \parallel \to 0$ which means $\mathcal{F}(\mathcal{B}_\iota)$ is equicontinuous.

As consequence of steps (1) to (3) together with the Arzela Ascoli theorem, one can get $\mathcal{F}: \mathcal{B}_\iota \to \mathcal{B}_\iota$ is completely continuous.

Step 4*:* Consider the following set of solutions of the system (1)

$$\mathcal{S} = \{x \in \mathcal{C}(\mathcal{J},\mathcal{X}): there\ exists\ \lambda \in [0,1]\ such\ that\ x = \lambda\mathcal{F}x\}.$$

We shall prove that $\mathcal{S}$ is bounded in $\mathcal{C}(\mathcal{J},\mathcal{X})$. For $x \in S$ and $\lambda \in [0,1]$, we have

$$\parallel x(t) \parallel = \parallel \lambda\mathcal{F}x(t) \parallel \leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} \parallel \xi(s,x(s)) \parallel ds$$
$$+\frac{t}{T\Gamma(q)}\int_0^T (T-s)^{q-1} \parallel \xi(s,x(s)) \parallel ds + (\frac{t}{T} - 1) \parallel \eta(x) \parallel + \frac{t}{T}x_0$$
$$\leq \frac{t(t^{q-1} + T^{q-1})}{\Gamma(q+1)}[\delta_1 \parallel \iota \parallel^{q_1} + \delta_2] + (\frac{t}{T} - 1)[\delta_3 \parallel \iota \parallel^{q_2} + \delta_4] + \frac{t}{T}x_0$$

The above inequality together with $q_1, q_2 \in [0,1)$ and step(2) show that $\mathcal{S}$ is bounded in $\mathcal{C}(\mathcal{J},\mathcal{X})$. As a consequence of Schaefer's fixed point theorem, we conclude that $\mathcal{F}$ has a fixed point which is the solution of the BVP (1).

Remark 3.1. *If $(H1) - (H5)$ hold and $\mathcal{F}: \mathcal{C}(\mathcal{J},\mathcal{X}) \to \mathcal{C}(\mathcal{J},\mathcal{X})$ is a linear operator then the set of solutions of the fractional BVP(1) is convex.*

Lemma 3.2. *The operator $\mathcal{F}: \mathcal{C}(\mathcal{J},\mathcal{X}) \to \mathcal{C}(\mathcal{J},\mathcal{X})$ is compact. Consequently, $\mathcal{F}$ is $\alpha -$ Lipschitz with zero constant.*

Proof: Consider a bounded subset $\mathcal{M} \subseteq \mathcal{C}(\mathcal{J},\mathcal{X})$. As we prove in Theorem 3.2 $\mathcal{F}: \mathcal{C}(\mathcal{J},\mathcal{X}) \to \mathcal{C}(\mathcal{J},\mathcal{X})$ is continuous and completely continuous. By applying the Arzela

Ascoli Theorem $\mathcal{F}(\mathcal{M})$ is a relatively compact subset of $\mathcal{C}(\mathcal{J},\mathcal{X})$. Hence, $\mathcal{F}:\mathcal{C}(\mathcal{J},\mathcal{X}) \to \mathcal{C}(\mathcal{J},\mathcal{X})$ is compact. Consequently, by Proposition 2.2 $\mathcal{F}$ is α −Lipschitz with zero constant.

Example 3.1. *Consider the following fractional BVP*

$$\begin{cases} {}^{c}D^{\frac{1}{2}}x(t) = \frac{(1-t^2)x(t)}{2\sqrt{\pi}} \quad t \in \mathcal{J} := [0,1],\ \ 0 < q < 1 \\ \qquad\qquad x(0) = \frac{x}{4},\quad x(1) = 0, \end{cases} \tag{9}$$

Set $q = \frac{1}{2}$, for $(t,x) \in [0,1] \times [0,+\infty)$. We have $\xi(t,x) = \frac{(1-t^2)x(t)}{2\sqrt{\pi}}$. By assumption (H1)-(H3), we can arrive

$$\begin{aligned} |\xi(t,x) - \xi(t,y)| &\le \frac{|1-t^2|}{2\sqrt{\pi}}|x(t) - y(t)| \\ &\le \frac{|1-t^2|}{2\sqrt{\pi}}|x-y|, \quad t \in [0,1] \\ &\le \frac{1}{2\sqrt{\pi}}|x-y| \Rightarrow \delta_\xi = \frac{1}{2\sqrt{\pi}} \end{aligned}$$

and,

$$|\eta(x) - \eta(y)| \le \frac{1}{4}|x-y| \Rightarrow \delta_\eta = \frac{1}{4}.$$

If $q = \frac{1}{2}$, we have $\Gamma(q+1) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$,

$$|\frac{2\delta_\xi T^q}{\Gamma(q+1)} + \delta_\eta| = |\frac{2(\frac{1}{2\sqrt{\pi}})}{\frac{1}{2}\sqrt{\pi}} + \frac{1}{4}| = \frac{2}{\pi} + \frac{1}{4} < 1$$

Thus, all assumptions in Theorem (3.1) are satisfied, our results can be used to solve the BVP (9).

4. Conclusion

We confirmed some sufficient conditions for the existence and uniqueness of a solution to BVP(1). We based on the fixed point theorem besides to topological technique of approximate solutions. Additionally, we studied some topological properties for the set of solutions. In the end, an example was presented to justify our results.

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