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# The Existence, Uniqueness and Continuity of a Solution to Mixed Fractional Constant Elasticity of Variance Model with Double Stochastic Volatilities

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***Abstract.*** In this work, we study the existence, uniqueness, continuity and some estimates of the solution to the stochastic differential equation with double stochastic volatilities driven by the mixed fractional Brownian motion.

***Keywords:*** Mixed fractional constant elasticity of variance model; Double stochastic volatilities; Existence; Uniqueness; Continuity.

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## 1. Introduction

Empirical evidence showed that the volatility smile observed in the financial market does not allow a constant volatility in financial models [1,2]. To model the volatility smile effectively one solution is to use the stochastic volatility under two cases. Either the function of stochastic processes is used to describe the volatility [3], or the additional Brownian motion is introduced to describe the stochastic parts of stochastic volatility (SV) models. In this paper, we focus on the second case.

Hull and White in [4] first introduced an SV model called Heston model in which the volatility of the market follows a mean-reverting Cox-Ingersoll-Ross process. The theoretical development of the SV model was introduced in [5] by studying the following equations

$$\begin{aligned} \mathrm{d}S(t) &= rS(t)\mathrm{d}t + \sqrt{v(t)}S(t)\mathrm{d}B_1(t) + \sigma S(t)\mathrm{d}J(t), \\ \mathrm{d}v(t) &= \kappa(\theta - v(t))\mathrm{d}t + \sigma_0(v(t)\mathrm{d}B_2(t), \end{aligned} \tag{1.1}$$

whose stochastic parts added a Levy process $\{J(t), t \quad 0\}$. Here $r, \kappa, \theta$, $\sigma$ and $\sigma_0$ are constants, $B_1(t)$ and $B_2(t)$ are standard Brownian motions with the assumption that $B_1(t)$, $B_2(t)$ and $J(t)$ are mutually independent. The paper also studied the existence and uniqueness of a strong solution to (1.1). Certain $L^p$ estimates of (1.1) were proved in [6].

However, all the existing SV models mentioned above are based on the Brownian motion, in which the increments follow the independent norm distribution. Many works argue that the returns of risky assets have long-range dependence properties, which are expressed by the increment of financial models. Using the Brownian motion to express the stochastic parts without considering its dependency to the financial modeling may have some serious disadvantages [7].

Recently, studying stochastic partial functional differential equation driven by fractional Brownian motion becomes a hot research topic. Fractional Brownian motion (*fBm*) is used to describe the stochastic parts of risky assets models because the increments of the fractional Brownian motion have the self-similar and long-range dependent properties. We refer readers to [7,8,9] for the motivation and references concerning the study of the fractional Brownian motion.

While the normality of increments assumption does not hold exactly, the pricing of options has been studied recently as alternative diffusion models. Specifically, some researches [12,13] focused on the constant elasticity of variance model called CEV model:

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)^{\alpha}\mathrm{d}B(t),$$

where $\alpha$ is the elasticity constant with $0 < \alpha < 1$. The model is better than Black-Scholes model since it captures the implied volatility smile (or skew phenomena) that the classical Black-Scholes model does not.

In spirit of *fBm* and CEV models, this paper uses mixed fractional Brown motion (*mfBm*), which is a linear combination of the Brown motion and fractional Brown motion to drive the following stock price equation of the CEV model

$$dS(t) = rS(t)dt + \sqrt{v_1(t)}S(t)^{\alpha_1}\mathrm{d}M_{1,1}^H(t) + \sqrt{v_2(t)}S(t)^{\alpha_2}\mathrm{d}M_{1,2}^H(t), \tag{1.2}$$

where the variance processes $v_1(t)$ and $v_2(t)$ are driven by another *mfBms* satisfy

$$dv_1(t) = \kappa_1(\theta_1 - v_1(t))\mathrm{d}t + \sigma_1 v_1(t)^{\beta_1}\mathrm{d}M_{2,1}^H(t), \tag{1.3}$$

$$dv_2(t) = \kappa_1(\theta_2 - v_2(t))\mathrm{d}t + \sigma_2 v_2(t)^{\beta_2}\mathrm{d}M_{2,2}^H(t), \tag{1.4}$$

$$\mathrm{d}M_{1,1}^H(t) \quad \mathrm{d}M_{1,2}^H(t) \qquad \mathrm{d}M_{2,1}^H(t) \quad \mathrm{d}M_{2,2}^H(t) \qquad 0,$$

$$\mathrm{d}M_{1,1}^H(t) \quad \mathrm{d}M_{2,2}^H(t) \qquad \mathrm{d}M_{1,2}^H(t) \quad \mathrm{d}M_{2,1}^H(t) \qquad 0,$$

$$\mathrm{d}M_{1,1}^H(t)\cdot \mathrm{d}B_{2,1}^H(t) = \rho_1 \mathrm{d}t^{2H}, \mathrm{d}M_{1,2}^H(t)\cdot \mathrm{d}M_{2,2}^H(t) = \rho_2 \mathrm{d}t^{2H},$$

where $v_1(0)$, $v_2(0)$ and $S(0)$ are given positive values, the non negative constants $\theta_i$ , $\sigma_i$ and $\kappa_i$ represent the long variance, the volatility of variance process and the rate at which $v_i$ reverts to $\theta_i, i = 1,2$ , respectively. $M_{1,1}^H(t), M_{1,2}^H(t), M_{2,1}^H(t)$ and $M_{2,2}^H(t)$ are *mfBm* processes whose concepts and relative conclusions will be given later. *r* is a constant interest rate, $\alpha_i$ and $\beta_i$ are elastic constants to stock price $S(t)$ and volatility of variance $v_i(t)$, for each $i = 1,2$, with the restriction that $\alpha_1 \vee \alpha_2 \le 1$ , $\beta_1 \vee \beta_2 \le 1$.

The main goal of this work is to investigate the existence, uniqueness and continuity of solutions to the dynamic model (1.2)-(1.4). The existence and uniqueness are analyzed in Section 2. Section 3 studies the continuity of the solution to the dynamic model (1.2)-(1.4).

## 2. Some preliminaries with respect to the mixed fractional Brown motion

The mixed fractional Brownian motion is a critical stochastic process and plays an important role in financial modeling. For a better understanding of the rest paper, we briefly review some basic concepts and properties of the mixed fractional Brownian motion.

### 2.1. Mixed fractional Brownian motion

Assume $H$ is a constant belonging to $(0,1)$. A fractional Brownian motion (*fBm*) $\left\{B^H(t), t \quad 0\right\}$ with Hurst parameter $H$ is a continuous and centered Gaussian process with covariance

$$E[B^H(t)B^H(s)] = \frac{1}{2}(t^{2H} + t^{2H} - \left|t-s\right|^{2H}) \text{ for any } s,t>0.$$

When $H = \frac{1}{2}$, the *fBm* becomes a standard Brownian motion denoted by $\{B(t), t \geq 0\}$.

A mixed fractional Brownian motion $\left\{M^H(t), t \quad 0\right\}$ is a linear combination of Brownian motion and fractional Brownian motion, defined with a filtered probability space $(\Omega, F, F_t, P)$ by:

$$M^H(t) = \lambda B(t) + B^H(t),$$

where $\lambda$ is a real constant, $P$ is the physical probability measure, and $\{F, t \geq 0\}$ denotes the $P$-augmentation of the filtration generated by $(B(t), B^H(t))$. A *mfBm* $\{M^H(t), t>0\}$ has the following properties[12,13,14,15] :

1. $M^H(0)=0$ and $E\left[M^H(t)\right]=0$ for any $t \quad 0$;
2. $M^H(t)$ is a centered Gaussian process and not a Markovian process for all $H \in (0,1)$;
3. $\{M^H(t), t \geq 0\}$ has homogeneous increments, $i,e, M^H(t+s) - M^H(s)$ has the same distribution as $B^H(t)$ for any $s,t \geq 0$;
4. The covariation functions of $M^H(t)$ and $M^H(s)$ are given by
$$E[M^H(t)M^H(s)] = \lambda^2 \cdot s \wedge t + \frac{1}{2}(t^{2H} + t^{2H} - \left|t-s\right|^{2H}) \text{ for any } s,t>0.$$
5. The increments of $M^H(t)$ are positively correlated if $\frac{1}{2} < H < 1$, uncorrelated if $H = 0.5$, and negatively correlated if $0 < H < \frac{1}{2}$.

*2.2. Basic spaces*

To study our problems, we introduce some new function spaces.

**Definition 2.1.** For any $s<t$, suppose that $C^\infty([s,t])$ denotes the Banach space of continuous functions equipped with the supremum norm $\|f\|_{s,t,\infty}$,

$$\left\|f\right\|_{s,t,} \quad \sup\left\{\left|f(r)\right|, s \quad r \quad t\right\}.$$

The space of Holder continuous functions of order $\beta > 0$ is denoted by $C^{\beta}([s,t])$, and its norm is

$$\left\|f\right\|_{s,t,\lambda} \quad \sup\{\frac{f(u)-f(v)}{\left|u-v\right|^{\lambda}}, s \quad v \quad u \quad t\}.$$

Let $X_0(t) = x$, and for $n = 1,2,\ldots$, we define the Picard iterations as

$$X_{n+1}(t,x) = x + \int_0^t \mu(X_n(s,x))\mathrm{d}s + \int_0^t \sigma(X_n(s,x))\mathrm{d}M^H(s),$$

for all $t \in [0,T]$. The integral wrt the Wiener process $\{B(t), t \geq 0\}$ is described as the Ito integral, while the integral wrt the process $\left\{B^H(t), t \quad 0\right\}$ is described as the wick integral. Let $D^m X$ denote the *m*-order derivatives of *X* w.r.t. *x*. It is easy to see that $D^m(X_{n+1})(t,x)$ for any integer $m \quad 0$ is continuous because $X_0$ is continuous in $(t,x)$. If we could prove $X_n$ converges in $C \quad (R)$, then the solution *X* of equation is $C^{\infty}$ in $x \in R$.

Let $M_{p,T}$ denote the set of all continuous $F_t$ adapted processes for $t \in [0,T]$ such that

$$E\left[\sup_{t\in[0,T]}|X_t|^p\right] < \infty$$

for a given $p \quad 2$. Then $M_{p,T} = L^p(R, C([0,T]))$ is a Banach space under the norm

$$\|X\|_{p,T} = \|X\|_{p,T(X)} = (E\left[\sup_{t\in[0,T]}|X(t,s)|^p\right]^{1/p}.$$

Let $N_{p,T}$ denote the set of locally integrable and measurable maps

$$\left\|X\right\|_{p,T}(\,) : x \quad M_{p,T}.$$

Thus, its Lebesgue integral is

$$\|X\|_{p,T,B}^p = \int_B \|X\|_{p,T}(x)\mathrm{d}x.$$

## 3. The existence and uniqueness

In this section, we prove the existence and uniqueness of the solution for the mixed Heston model by extending the idea of [16] for the mixed stochastic differential equation.

**Theorem 3.1.** For each $i = 1,2$, the volatility equation of the mixed CEV model has a unique solution $v_i(t)$ where $t \in [0,T]$.

**Proof:** Here, we confirm the solution's existence and uniqueness for stochastic equations (1.3) and (1.4). Define an operator *A* in $N_{p,T}$ as follows

$$(Av)(t,v_0) = v_0 + \int_0^t \kappa(\theta - v(s,v_0))\mathrm{d}s + \int_0^t \sigma v(s,v_0)^{\beta}\mathrm{d}M^H(s) \qquad (3.1)$$

for $v \in N_{p,T}$ , and it suffices to prove that the operator $A$ has a unique fixed point $v_\infty \in N_{p,T}$.

First, we prove the existence of fixed point: $Av$ is an element of $N_{p,T}$ for each $v(t,v_0)$ in $N_{p,T}$ . We can see that $Av$ is continuous in $x \in R$. Using

$$(a+b+c)^n \le 3^{n-1}(a^n+b^n+c^n),$$

we have

$$\left|(Av)(s,v_0)\right|^p \le 3^{p-1}\left|v_0\right|^p + 3^{p-1}\left|\int_0^t \kappa(\theta - v(s,v_0))\mathrm{d}s\right|^p + 3^{p-1}\left|\int_0^t \sigma v(s,v_0)^\beta \mathrm{d}M^H(s)\right|^p.$$

Then

$$\left\|Av(t,v_0)\right\|_{p,T}^p \quad 3^{p-1}E\ \left|v_0\right|^p \quad 3^{p-1}E\ M_1 \quad 3^{p-1}E\ M_2\ , \tag{3.2}$$

Where

$$M_1 = \left|\int_0^t \kappa(\theta - v(s,v_0))\mathrm{d}s\right|^p, M_2 = \left|\int_0^t \sigma v(s,v_0)^\beta \mathrm{d}M^H(s)\right|^p.$$

Now, we compute $E[M_1]$ and $E[M_2]$. Using $(a \quad b)^n \quad 2^{n-1}(a^n \quad b^n)$ and Holder inequality, we have

$$E[M_1] \le \kappa^p t\int_0^t \left|\theta - v(s,v_0)\right|^p \mathrm{d}s \le 2^{p-1}\kappa^p\theta^p t^2 + 2^{p-1}\kappa^p t\int_0^t E\left[\left|v(s,v_0)\right|^p\right]\mathrm{d}s. \tag{3.3}$$

Noting that $M^H(t) = \lambda B(t) + B^H(t)$ and using $(a+b)^n \le 2^{n-1}(a^n+b^n)$ to $M_2$,we have

$$E[M_2] \le 2^{p-1}\lambda^p E[M_3] + 2^{p-1}E[M_4], \tag{3.4}$$

where

$$M_3 = \left|\int_0^t \sigma v(s,v_0)^\beta \mathrm{d}B(s)\right|^p, M_4 = \left|\int_0^t \sigma v(s,v_0)^\beta \mathrm{d}B^H(s)\right|^p.$$

Using the B-D-G inequality and fractional B-D-G inequality[15] to $E\ M_3$ and $E[M_4]$, respectively, and using Holder inequality, we obtain

$$E[M_3] \le \sigma^p\lambda^p\left|\int_0^t E\left[v(s,v_0)^{2\beta}\right]\mathrm{d}s\right|^{\frac{p}{2}} \le \sigma^p\lambda^p t\int_0^t E\left[v(s,v_0)^p\right]\mathrm{d}s, \tag{3.5}$$

$$\begin{aligned} E\ M_4 &\quad \sigma^p H^{\frac{p}{2}} t^{pH-\frac{P}{2}}\left|\ \int_0^t E\ v(s,v_0)^{2\beta}\ \mathrm{d}s\right|^{\frac{p}{2}} \\ &\quad \sigma^p H^{\frac{p}{2}} t^{pH+1-\frac{P}{2}} \int_0^t E\ v(s,v_0)^p\ \mathrm{d}s. \end{aligned} \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we have

$$E[M_2] \le C_1(\lambda,p,\sigma,T,H)\int_0^t E\left[v(s)^p\right]\mathrm{d}s, \tag{3.7}$$

where $C_1(\lambda,p,\sigma,T,H) = 2^{p-1}\lambda^p\sigma^p\lambda^p T + 2^{p-1}\sigma^p H^{\frac{p}{2}} T^{pH+1-\frac{P}{2}}$. Consequently, we substitute (3.7) and (3.3) into (3.2) to arrive at

$$\|Av\|^p_{p,T(v_0)} \le 3^{p-1}E\left[|v_0|^p\right] + 6^{p-1}\kappa^p\theta^p T^2 + C_2(\lambda,p,\sigma,\kappa,T,H)\int_0^T E\left[|v(s,v_0)|^p\right]\mathrm{d}s \quad (3.8)$$

uniformly in $v_0$ on each bounded domain $B \subset R$, where

$$C_2(\lambda,p,\sigma,\kappa,T,H) = 6^{p-1}\kappa^p\theta^p T^2 + 3^{p-1}C_1(\lambda,p,\sigma,T,H).$$

Thus, we have $Av \in N_{p,T}$.

Now we prove the uniqueness of the fixed point. Suppose $Y(t,v_0)$ and $Z(t,v_0)$ are fixed points in $N_{p,T}$. Therefore, we have

$$\left|AY(t,v_0) - AZ(t,v_0)\right|^p$$
$$\left|\kappa \int_0^t Y(s,v_0) \quad Z(s,v_0)\mathrm{d}s \quad \sigma_1 \int_0^t Y(s,v_0)^{\beta_1} \quad Z(s,v_0)^{\beta_1}\mathrm{d}M^H(s)\right|^2.$$

Using $(a+b)^p \le 2^{p-1}(a^p+b^p)$, we obtain

$$\left|AY(t,v_0) - AZ(t,v_0)\right|^p$$
$$\le 2^{p-1}|\kappa|^p\left|\int_0^t Y(s,v_0) - Z(s,v_0)\mathrm{d}s\right|^p + 2^{p-1}|\sigma|^p\left|\int_0^t Y(s,v_0)^{\beta_1} - Z(s,v_0)^{\beta_1}\mathrm{d}M^H(s)\right|^p.$$

Then

$$\|AY(t,v_0) - AZ(t,v_0)\|^p_{p,T} \le C_3(P)2^{P-1}|\kappa|^P M_5 + C_3(P)2^{P-1}|\sigma|^P M_6, \quad (3.9)$$

where

$$M_5 = E\left|\int_0^t Y(s,v_0) - Z(s,v_0)\mathrm{d}s\right|^p,$$

$$M_6 = E\left|\int_0^t Y(s,v_0)^{\beta_1} - Z(s,v_0)^{\beta_1}\mathrm{d}M^H(s)\right|^p.$$

Following the similar proof of (3.3) and (3.7), we obtain

$$M_5 \le T\int_0^t E\left[|Y(s,v_0) - Z(s,v_0)|^p\right]\mathrm{d}s, \quad (3.10)$$

$$M_6 \le C_4(\lambda,p,\sigma,T,H)\int_0^T E\left[|Y(s,v_0) - Z(s,v_0)|^p\right]\mathrm{d}s. \quad (3.11)$$

Substituting (3.10) and (3.11) into (3.9) yields

$$\|AY - AZ\|^p_{p,T} \le C_5(\lambda,p,\sigma,\kappa,T,H)\int_0^T E\left[|Y(s,v_0) - Z(s,v_0)|^p\right]\mathrm{d}s,$$

where

$$C_5(\lambda,p,\sigma,\kappa,T,H)$$
$$= C_3(P)2^{P-1}|\kappa|^P T + C_3(P)2^{P-1}|\sigma|^P C_4(\lambda,p,\sigma,T,H)$$
$$= 2^{P-1}C_3(P)|\kappa|^P T + 4^{P-1}C_3(P)\sigma^{2p}\lambda^{2p}T + 4^{P-1}C_3(P)\sigma^{2p}H^{\frac{p}{2}}T^{pH+1-\frac{p}{2}}$$

Therefore

$$\|AY - AZ\|^p_{p,T} \le C_5(\lambda,p,\sigma,\kappa,T,H)T\|Y - Z\|^p_{p,T}$$

uniformly in $v_0$ on each bounded domain $B$ . Since

$$C_5(\lambda, p, \sigma, \kappa, T_0, H)T_0 < 1$$

with be enough for small $T_0$ , it indicates that $A: N_{p,T} \to N_{p,T}$ is a contraction with $T = T_0$ . Therefore, by the contraction mapping principle, $A$ has a unique fixed point $v \quad N_{p,T}$ . Moreover, let $v \quad N_{p,T}$ be arbitrary, $v_n = A^n v_0$ . The sequence $\{v_n,$ n=1, 2, . . .$\}$ converges to $v_\infty \in N_{p,T}$ for all $p \geq 2$ . Consequently, the uniqueness of the solution is proved choosing $v = v$ .

Next, we will derive $L_p$ estimate for the solution of the volatility equations.

**Lemma 3.1.** Let $T > 0$ be fixed. Then for any positive constant $C_i = C(\lambda, p, \sigma, \kappa_i, H, v_i(0), T)$, we have

$$E \sup_{t \quad 0,T} \left|v_i(t, v_i(0))\right|^p \quad C_i, i \quad 1,2. \tag{3.12}$$

**Proof:** Here we only prove the case of $i = 1$. For any $t \in [0, T]$, we have

$$v_1(t, v_1(0)) = v_1(0) + \int_0^t \kappa_1(\theta_1 - v_1(s, v_1(0)))\mathrm{d}s + \sigma_1 \int_0^t v_1(s, v_1(0))^{\beta_1} \mathrm{d}M_{2,1}^M(s)$$.

First, we consider the case that $p \geq 2$. Using the Young s inequality, we have for any $p \quad 2$ that

$$\left|v_1(t, v_1(0))\right|^p \leq 3^{P-1}(\left|v_1(0)\right|^p + M_7 + M_8), \tag{3.13}$$

where

$$M_7 = \kappa_1^p \left|\int_0^t \theta - v_1(s, v_1(0))\mathrm{d}s\right|^p, \quad M_8 = \sigma_1^p \left|\int_0^t v_1(s, v_1(0))^{\beta_1} \mathrm{d}M_{2,1}^M(s)\right|^p.$$

Now, we compute $E \ M_7$ and $E[M_8]$ . Using the Holder inequality and $(a+b)^n \leq 2^{n-1}(a^n + b^n)$, we have

$$\begin{aligned} E \ M_7 \quad & 2^{p-1}\kappa_1^p\theta^p T \quad 2^{p-1}\kappa_1^p \left| \ _0^t v_1(s, v_1(0))\mathrm{d}s\right|^p \\ & 2^{p-1}\kappa_1^p\theta^p T \quad 2^{p-1}\kappa_1^p T \left| \ _0^t E \ \left|v_1(s, v_1(0))\right|^p \ \right| \mathrm{d}s. \end{aligned} \tag{3.14}$$

Following the similar proof of (3.7), we obtain

$$E[M_8] \leq C_6(\lambda, p, \sigma, T, H)\int_0^t E\left[\left|v_1(s, v_1(0))\right|^p\right]\mathrm{d}s. \tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13), and letting

$$C_7 = 3^{p-1}\left|v_1(0)\right|^p + 6^{p-1}\kappa_1^p\theta^p T, C_8 = 3^{p-1}C_1(\lambda, p, \sigma, T, H) + 6^{p-1}\kappa_1^p T$$

we obtain

$$E\left[\left|v_1(s,v_1(0))\right|^p\right] \le C_7 + C_8\int_0^t E\left[\left|v_1(s,v_1(0))\right|^p\right]\mathrm{d}s\,. \tag{3.16}$$

Hence, the Gronwall inequality implies that

$$\sup_{t\in[0,T]} E\left[\left|v_1(t)\right|^p\right] \le C_7\exp\{C_8T\} = C_9(\lambda,p,\sigma,\theta_1,\kappa_1 H,v_1(0),T),\ p\ge 2\,. \tag{3.17}$$

Second, we prove that (3.17) still holds for any $1 \le p < 2$. Using the Cauchy inequality, we obtain

$$E\left[\left|v_1(t,v_1(0))\right|^p\right] \le E\left[\left|v_1(t,v_1(0))\right|^{2p}\right]^{\frac{1}{2}} \le \left[\sup_{t\in[0,T]} E\left[\left|v_1(t,v_1(0))\right|^{2p}\right]\right]^{\frac{1}{2}}.$$

Noting that $2p \ge 2$ and using (3.17), we obtain

$$E\left[\left|v_1(t,v_1(0))\right|^p\right] \le \sqrt{C_9(\lambda,p,\sigma,\theta_1,\kappa_1 H,v_1(0),T)}.$$

Because $t\in[0,T]$ is arbitrary, (3.17) is proved when $1\le p<2$.

Third, if $0<p<1$, note that

$$\begin{aligned}
&\left|v_1(t,v_1(0))\right|^p \\
&=\left|v_1(t,v_1(0))\right|^p I_{\{|v_1(t,v_1(0))|\ge 1\}} + \left|v_1(t,v_1(0))\right|^p I_{\{|v_1(t,v_1(0))|<1\}} \\
&\le \left|v_1(t,v_1(0))\right|^{p+1} I_{\{|v_1(t,v_1(0))|\ge 1\}} + \left|v_1(t,v_1(0))\right|^p I_{\{|v_1(t,v_1(0))|<1\}}.
\end{aligned}$$

Further we have

$$\left|v_1(t,v_1(0))\right|^p \le \left|v_1(t,v_1(0))\right|^{p+1} I_{\{|v_1(t,v_1(0))|\ge 1\}} + 1 \le \left|v_1(t,v_1(0))\right|^{p+1} + 1.$$

Hence it follows from the case $1<p<2$

$$\sup_{t\in[0,T]} E\left[\left|v_1(t,v_1(0))\right|^p\right] \le \sqrt{C_9(\lambda,p,\sigma,\theta_1,\kappa_1,H,v_1(0),T)} + 1.$$

This completes the proof of Lemma 3.1.

Following the proof of Theorem 3.1 and Lemma 3.1, we can prove the following lemma for the stock price equation.

**Lemma 3.2.** Stock price equation of the CEV model has a unique solution. Moreover

$$\sup_{t\in[0,T]} E\left[\left|S(t)\right|^p\right] \le C_{10}(\mu,\lambda,p,\sigma,H,\theta_1,\theta_2,\kappa_1,\kappa_2,v_1(0),v_2(0),S(0),T)\,. \tag{3.18}$$

## 4. Continuity

In this section, we discuss the continuity of the stock price equation of the CEV model.

**Theorem 4.1.** Stock price process of the CEV model $\{S(t), t>0\}$ is continuous in $t$.

**Proof:** Note that for any $0 \le s \le t \le T$,

$$\begin{aligned}
&S(t)-S(s) \\
&= \int_s^t \mu S(s)\mathrm{d}s + \int_s^t \sqrt{v_1(\tau)}S(\tau)^{\alpha_1}\mathrm{d}M_{1,1}^H(\tau) + \int_s^t \sqrt{v_2(\tau)}S(\tau)^{\alpha_1}\mathrm{d}M_{1,2}^H(\tau).
\end{aligned}$$

Using $(a+b+c)^4 \le 3^3(a^4+b^4+c^4)$, we obtain

$$\left|S(t)-S(s)\right|^4 \le 3^3A_3 + 3^3A_4 + 3^3A_5, \tag{4.1}$$

where

$$A_3=\left|\int_s^t \mu S(s)\mathrm{d}s\right|^4,\ \ A_4=\left|\int_s^t \sqrt{v_1(\tau)}S(\tau)^{\alpha_1}\mathrm{d}M_{1,1}^H(\tau)\right|^4, A_5=\left|\int_s^t \sqrt{v_2(\tau)}S(\tau)^{\alpha_1}\mathrm{d}M_{1,2}^H(\tau)\right|^4.$$

It follows the Cauchy inequality,

$$\left|\int_s^t S(s)\mathrm{d}s\right|^4 \le (t-s)^{\frac{3}{4}}\int_s^t \mu S(s)\mathrm{d}s.$$

Therefore,

$$E[A_3]\le|\mu|^4(t-s)^{\frac{3}{4}}E\left[\int_S^t|S(s)|^4\mathrm{d}s\right]\le|\mu|^4(t-s)^{\frac{3}{4}}\int_s^t\sup_{t\in[0,T]}E\left[|S(s)|^4\right]\mathrm{d}s. \tag{4.2}$$

(3.18) and (4.2) imply that

$$E[A_3]\le|\mu|^4C_{11}|t-s|^{\frac{7}{4}}. \tag{4.3}$$

Now we pay attention to $E[A_4]$ and $E[A_5]$. Using the B-D-G inequality[12,14] , we obtain

$$E[A_4]\le\left(\lambda^2+2HT^{2H-1}\right)^2\left|\int_s^t E\left[|v_1(\tau)|\cdot|S(\tau)|^{2\alpha_1}\right]\mathrm{d}\tau\right|^2, E[A_5]\le\left(\lambda^2+2HT^{2H-1}\right)^2\left|\int_s^t E\left[|v_2(\tau)|\cdot|S(\tau)|^{2\alpha_2}\right]\mathrm{d}\tau\right|^2.$$

We use the Holder inequality to arrive at

$$E[A_4]\le\left(\lambda^2+2HT^{2H-1}\right)^2\int_s^t E\left[|v_1(\tau)|^2\right]d\tau\int_s^t E\left[|S(\tau)|^{4\alpha_1}\right]\mathrm{d}\tau,$$

$$E[A_5]\le\left(\lambda^2+2HT^{2H-1}\right)^2\int_s^t E\left[|v_2(\tau)|^2\right]d\tau\int_s^t E\left[|S(\tau)|^{4\alpha_2}\right]\mathrm{d}\tau.$$

It follows (2.9) and (2.15) that

$$E[A_4]\le C_{12}|t-s|^2, E[A_5]\le C_{13}|t-s|^2. \tag{4.4}$$

Combing (4.1), (4.3) and (4.4), we yield

$$E\left[|S(t)-S(s)|^4\right]\le C_{13}|t-s|^2+|\mu|^4C_{12}|t-s|^{\frac{7}{4}}\le\left(C_{13}T^{\frac{1}{4}}+|\mu|^4C_{12}\right)\cdot|t-s|^{\frac{7}{4}}. \tag{4.5}$$

Therefore, the theorem is proved.

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