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Short Communication

On the Diophantine Equation $11 \cdot 3^x + 11^y = z^2$ where x, y and z are Non-Negative Integers

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Abstract. In this article, we find non-negative solutions of the Diophantine equation $11 \cdot 3^x + 11^y = z^2$ where x, y and z are non-negative integers. The basic theorems, which involve divisible, are applied to prove all solutions. The result indicates that the equation has the unique solution $(x, y, z) = (2,0,10)$.

Keywords: Diophantine equation; factoring method; modular arithmetic method

AMS Mathematics Subject Classification (2010): 11D45

1. Introduction

Over three hundred years, Diophantine equation has been known and studied by mathematicians. There are three classic problems including "is the equation solvable?", "is the number of its solution finite or infinite?" and "how is all of it solution determined?". In the last decade, the Diophantine equations in the form $a^x + b^y = z^2$ have been studied by many mathematicians. Some articles used theorems in Number theory to solve the equations [3, 4, 6, 8, 14]. Some articles applied Catalan's conjecture to solve the equations [1, 2, 5, 7, 9, 12, 13, 15, 16]. However, there is no general method for finding a solution.

In 2021, Komon and Pailin [10] proved that the equation $17^x + 83^y = z^2$ and $29^x + 71^y = z^2$ have same unique non-negative integer solution, $(x, y, z) = (1,1,10)$. In the same year, Singha [11] solved the solution of the equation $8^x + p^y = z^2$ where p is odd prime number and $p \not\equiv 1 \pmod 8$. Recently, a number of mathematicians are still studying and finding the solution of a new equation. This topic is great important in contemporary mathematics.

In this work, we use knowledge in Number theory such as the factoring method and the modular arithmetic method to find all solutions of the Diophantine equation

$11 \cdot 3^x + 11^y = z^2$ where x, y and z are non-negative integers.

2. Main results

Theorem 2.1. $(x, y, z) = (2,0,10)$ is the unique solution to the Diophantine equation

$$11 \cdot 3^x + 11^y = z^2$$

where x, y and z are non-negative integers.

Proof: Let x, y and z be non-negative integers such that

$$11 \cdot 3^x + 11^y = z^2. \tag{1}$$

We first consider four cases.

Case 1: $x = 0$ and $y = 0$. We have $z^2 = 11 + 1 = 12$. This is impossible.

Case 2: $x = 0$ and $y > 0$. We have $11 + 11^y = z^2$ or

$$11\left(1 + 11^{y-1}\right) = z^2. \tag{2}$$

From (2), we separate y into $y = 1$ and $y > 1$. If $y = 1$, then $z^2 = 22$. This is impossible. If $y > 1$, then $11 | z^2$. There is some positive integer k such that $z = 11k$. Then $11\left(1 + 11^{y-1}\right) = \left(11k\right)^2$ or $1 = 11\left(k^2 - 11^{y-2}\right)$. This is impossible.

Case 3: $x > 0$ and $y = 0$. From (1), we have $11 \cdot 3^x + 1 = z^2$ or

$$11 \cdot 3^x = (z-1)(z+1). \tag{3}$$

This implies that $11 | (z-1)(z+1)$. This means that $11 | z-1$ or $11 | z+1$.

If $11 | z-1$, then we have $r \in {}^{+} \cup \{0\}$ such that $z - 1 = 11 \cdot 3^r$. Therefore, $z + 1 = 11 \cdot 3^r + 2$. By (3), we have $11 \cdot 3^x = 11 \cdot 3^r \left(11 \cdot 3^r + 2\right)$ or

$$3^{x-r} = 11 \cdot 3^r + 2. \tag{4}$$

We consider when $r = 0$ and $r > 0$. If $r = 0$, then $3^x = 13$. This is impossible. If $r > 0$, it follow that $x - r > 0$. Thus $3^{x-r} \equiv 0 \pmod 3$ and $11 \cdot 3^r \equiv 0 \pmod 3$. Form (4), we must have $2 \equiv 0 \pmod 3$, impossible.

If $11 | z+1$, then $11 \nmid z-1$. From (3), there exist $r, s \in {}^{+} \cup \{0\}$ such that $z + 1 = 11 \cdot 3^r$ and $z - 1 = 3^s$ where $r + s = x$. Hence

$$11 \cdot 3^r - 3^s = 2 \tag{5}$$

From (5), if $s = 0$, then we have $11 \cdot 3^r = 3$ or $11 \cdot 3^{r-1} = 1$. This is impossible. If $r = 0$, then we have $3^s = 9$. This gives $s = 2$, so we obtain $x = 2$. Hence we obtain $x = 2$, $y = 0$ and $z = 10$. They are the solution of this Diophantine equation.

If $s > 0$ and $r > 0$, then we can see that $11 \cdot 3^r \equiv 0 \pmod 3$ and $3^s \equiv 0 \pmod 3$. Therefore $11 \cdot 3^r - 3^s \equiv 0 \pmod 3$, implying that $2 \equiv 0 \pmod 3$. This is impossible.

case 4: $x > 0$ and $y > 0$, this case yields

$$11\left(3^x + 11^{y-1}\right) = z^2 \quad . \tag{6}$$

It means that $11 | z^2$. This implies that $11 | z$. Thus, there exists $k \in {}^{+}$ such that $z = 11k$

We can write (6) to be

$$3^x + 11^{y-1} = 11k^2. \quad (7)$$

Because $y > 0$, we consider $y = 1$ and $y > 1$. If $y = 1$ then we obtain $11k^2 - 3^x = 1$. We note that $11k^2 \equiv -k^2 \pmod{3}$ and $3^x \equiv 0 \pmod{3}$, then $-k^2 \equiv 1 \pmod{3}$. It follows that $k^2 \equiv -1 \pmod{3}$, implying that $k^2 \equiv 2 \pmod{3}$. This is impossible. If $y > 1$ then it implies that $3^x \equiv 0 \pmod{11}$. This means that $11 \mid 3^x$. This is impossible. □

3. Conclusion

In this work, we have shown that the Diophantine equation $11 \cdot 3^x + 11^y = z^2$ has the unique solution where x, y and z are non-negative integers. In the proof, we consider four main cases including $x = 0$ and $y = 0$, $x = 0$ and $y > 0$, $x > 0$ and $y = 0$, and $x > 0$ and $y > 0$. The proof reveals that the unique solution is $(x, y, z) = (2, 0, 10)$.

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Authors' Contributions. All the authors contributed equally to this work.

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