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Short Communication

On the Diophantine Equation $7^x - 2^y = z^2$ where x, y and z are Non-Negative Integers

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Abstract. In this article, we prove all solutions of the exponential Diophantine equation $7^x - 2^y = z^2$ where x, y and z are non-negative integers. The mathematical principles are applied to obtain the solutions such as factoring method modular arithmetic method and Catalan's conjecture. The result reveals that there is only a trivial solution to the equation.

Keywords: Diophantine equation; factoring method; modular arithmetic method

AMS Mathematics Subject Classification (2010): 11D14

1. Introduction

Over a decade, many mathematicians have studied the exponential Diophantine equations. They studied that equations are solvable or how many solutions they are. In 2004, P. Mihailescu [7] proved that equation $a^x - b^y = 1$ where a, b, x and y are integers and $\min\{a, b, x, y\} > 1$ has only one solution that is $(a, b, x, y) = (3, 2, 2, 3)$ which is known as Catalan's conjecture. The Catalan's conjecture was applied to solve the Diophantine equations in the form $a^x + b^y = z^2$ where x, y and z are non-negative integers [1, 3-6, 8-10,13]. Meanwhile, some researchers studied the Diophantine equation in the form $a^x - b^y = z^2$ where x, y and z are non-negative integers. In 2019, the equation $2^x - 3^y = z^2$ where x, y and z are non-negative integers was proved that it has only three solutions [11]. After that, M. Buosi et al. [2] studied the equation $p^x - 2^y = z^2$. They focused on that $p = k^2 + 2$ is a prime number and $k \geq 0$. In 2021, the equation $7^x - 5^y = z^2$ was proved that $(x, y, z) = (0, 0, 0)$ is only one solution (trivial solution) to the equation [12]. According to previous studies, there are no general method to prove solutions to the classes of the equation in this form. The study of solutions to individual equations still need. In this paper, we show all solutions of the exponential Diophantine equation $7^x - 2^y = z^2$ where x, y and z are non-negative integers. We applied the mathematical knowledge to prove the equation such as factoring method modular arithmetic method and Catalan's conjecture.

2. Preliminary

Lemma 2.1. (Catalan's conjecture) [7] Let a, b, x and y are integers. The Diophantine equation $a^x - b^y = 1$ with $\min\{a, b, x, y\} > 1$ has the unique solution $(a, b, x, y) = (3, 2, 2, 3)$.

3. Main result

Theorem 3.1. The Diophantine equation $7^x - 2^y = z^2$ has the only one solution $(x, y, z) = (0, 0, 0)$ where x, y and z are non-negative integers.

Proof: Let x, y and z are non-negative integers such that

$$7^x - 2^y = z^2. \tag{1}$$

Now, we consider four cases as follows.

Case 1: $x = 0$ and $y = 0$. From (1), we have $z = 0$.

Case 2: $x = 0$ and $y > 0$. From (1), we have $1 - 2^y = z^2$. This is impossible because $1 - 2^y < 0$.

Case 3: $x > 0$ and $y = 0$. We have $7^x - 1 = z^2$ or

$$7^x - z^2 = 1. \tag{2}$$

We divide x into two subcases such that $x = 1$ and $x > 1$.

Subcase 3.1: If $x = 1$, then we obtain $7 - z^2 = 1$ or $z^2 = 6$. This is impossible.

Subcase 3.2: If $x > 1$, by Lemma 2.1, (2) has no solution.

Case 4: $x > 0$ and $y > 0$. We consider two subcases including $y = 1$ and $y \geq 2$.

Subcase 4.1: $y = 1$. From (1), we obtain

$$7^x - 2 = z^2. \tag{3}$$

This implies that z^2 is odd, so z is odd. It follows that $z^2 \equiv 1 \pmod 8$. From (3), we obtain

$$1 \equiv (-1)^x - 2 \pmod 8 \quad \text{or} \quad 3 \equiv (-1)^x \pmod 8.$$

This is impossible.

Subcase 4.2: $y \geq 2$, From (1), we have $z^2 \equiv (-1)^x \pmod 4$.

Since $z^2 \not\equiv -1 \pmod 4$, x must be even. We let $x = 2k$ for some $k \in {}^{+}$. (1) becomes $7^{2k} - 2^y = z^2$ or $2^y = (7^k - z)(7^k + z)$.

Then there are α and β which are integers such that $7^k - z = 2^\alpha$ and $7^k + z = 2^\beta$ where $0 \leq \alpha < \beta$ and $\alpha + \beta = y$. These yield $2^\alpha + 2^\beta = 2 \cdot 7^k$ or $2^\alpha \left(1 + 2^{\beta - \alpha}\right) = 2 \cdot 7^k$.

Then, $\alpha = 1$ and $1 + 2^{\beta-1} = 7^k$. We have

$$7^k - 2^{\beta-1} = 1. \tag{4}$$

If $\beta = 1$ then $7^k = 2$. This is impossible. If $\beta = 2$ then $7^k = 3$. This is impossible. If $\beta = 3$ then $7^k = 5$. This is impossible. If $\beta \geq 4$. By Lemma 2.1, (4) has no solution. Hence, the equation $7^x - 2^y = z^2$ has the only one solution $(x, y, z) = (0, 0, 0)$. □

4. Conclusion

We have proved all solutions of the Diophantine equation $7^x - 2^y = z^2$ where x, y and z are non-negative integers. We separate in four cases including case 1: $x, y = 0$, case 2: $x = 0$ and $y > 0$, case 3: $x > 0$ and $y = 0$, and case 4: $x > 0$ and $y > 0$. Finally, we can show that the equation has only a trivial solution $(x, y, z) = (0, 0, 0)$.

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Authors' Contributions. All the authors contributed equally to this work.

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