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Accurate and Connected Accurate Domination Polynomials of Graphs

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Abstract. Representation of a graph through polynomial equations has been done using different domination parameters. In this article, we introduce two domination polynomials called accurate domination polynomial (ADP) and connected accurate domination polynomial ($CADP$) of a graph G of order n. We denote ADP by $D_A(G,x)$ and $CADP$ by $D_{CA}(G,x)$. We obtain ADP and $CADP$ of some standard graphs.

Keywords: Domination, Accurate Domination, connected accurate domination, Accurate domination polynomial, connected accurate domination polynomial.

AMS Mathematics Subject Classification (2010): 05C30

1. Introduction

Graphs which are finite, non-trivial, undirected with neither loops nor multiple edges are taken into account in this paper. A Set D of a graph $G = (V, E)$ is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. For a survey on domination number $\gamma(G)$, we refer [1]. Accurate Dominating Set (ADS) is a dominating set such that $V - D$ has no dominating set of cardinality $|D|$. The accurate domination number $\gamma_a(G)$ is the minimum cardinality of an ADS. An ADS is said to be a Connected Accurate Dominating Set ($CADS$) if the induced sub graph $\langle D \rangle$ is connected. The connected accurate domination number $\gamma_{ca}(G)$ is the minimum cardinality of a $CADS$. Both the domination parameters used in this article are introduced by Kulli and Kattimani [2,3].

A graph G is a complete graph with n vertices if there is an edge between every pair of vertices. We denote a complete graph by K_n. A bigraph (or bipartite graph) G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex in V_1 with a vertex in V_2. If there is an edge between every vertex in V_1 with every vertex in V_2, then the bipartite graph is called a complete bipartite graph and is usually denoted by $K_{m,n}$ with $|V_1| = m$ and $|V_2| = m$. A star is a complete bipartite graph $K_{1,n}$. For $n \geq 4$, the wheel graph W_n is defined to be the graph $K_1 + C_{n-1}$, where C_{n-1} is a cycle with $n - 1$ vertices. A bi-star is a tree obtained from the graph K_2 with end vertices u and v by attaching m pendant edges to u and n pendant edges to v. A bi-star is denoted by $B(m, n)$. For terminologies and notations, we refer [5].

Domination polynomial was initiated by Arocha et al. [3] and was developed later by Alikhani and Yee-hock Peng [4]. Inspired by the work in [4], we introduce accurate domination polynomial and connected accurate domination polynomial of a graph.

2. *ADP* and *CADP* of a graph

In this section, we introduce two new polynomials called Accurate Domination Polynomial and Connected Accurate Domination Polynomial as follows:

Definition 2.1. Let $d_A(G, i)$ be the total number of ADS of cardinality i of a simple connected graph G. Then ADP , $D_A(G, x)$ of a graph G is defined as follows:

$$D_A(G, x) = \sum_{i=\gamma_a(G)}^{n} d_A(G, i)\, x^i$$

The roots of the ADP are called accurate domination roots and we denote them by $Z\big(D_A(G, x)\big)$.

Definition 2.2. Let $d_{CA}(G, i)$ be the total number $CADS$ of cardinality i of a simple connected graph G. Then $CADP$, $D_{CA}(G, x)$ of a graph G is defined as follows:

$$D_{CA}(G, x) = \sum_{i=\gamma_{ca}(G)}^{n} d_{CA}(G, i)\, x^i$$

The roots of the $CADP$ are called accurate domination roots and we denote them by $Z\big(D_{CA}(G, x)\big)$.

Example 2.3. consider a graph, path P_5 with 5 vertices as in Figure 1.

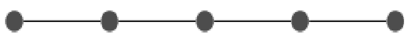

Figure 1

The accurate domination number of the above graph is 3. But P_5 has 8 ADS with 3vertices, 5 ADS with 4 vertices and 1 ADS with 5 elements. Thus $D_A(P_5, x) = 8x^3 + 5x^4 + x^5$.The accurate dominating roots are $Z\big(D_A(G, x)\big) = 0,0,0, -2.5 \pm 1.3228757\, i$.

We note that $\gamma_{ca}(P_5) = 3$. But P_5 has 1 $CADS$ with 3vertices, 2 $CADS$ with 4 vertices and 1 ADS with 5 elements. Thus $D_{CA}(P_5, x) = x^3 + 2x^4 + x^5$.The accurate dominating roots are $Z\big(D_{CA}(G, x)\big) = 0,0,0, -1, -1$.

3. Main results

Theorem 3.1. ADP of a complete graph K_n is

$$D_A(K_n, x) = x^n + nC_{n-1}x^{n-1} + nC_{n-2}x^{n-2} + \cdots + nC_{\gamma_a(K_n)}x^{\gamma_a(K_n)}$$

Proof: As $\gamma_a(G) = \lfloor n \div 2 \rfloor + 1$, any set of vertices with $< \gamma_a(G)$ elements will not be an ADS. Thus the minimum power of x in ADP will be $\gamma_a(G)$. Further we can select

nC_{n-1}number of ADS with $(n-1)$ number of vertices, nC_{n-2}number of ADS with $(n-2)$ number of vertices, and so on. The only ADS with n elements will be the vertex set of $K_n$itself.

Thus,

$$D_A(K_n, x) = x^n + nC_{n-1}x^{n-1} + nC_{n-2}x^{n-2} + \cdots + nC_{\gamma_a(K_n)}x^{\gamma_a(K_n)}$$

Theorem 3.2. $CADP$ of a complete graph K_n is

$$D_{CA}(K_n, x) = x^n + nC_{n-1}x^{n-1} + nC_{n-2}x^{n-2} + \cdots + nC_{\gamma_{ca}(K_n)}x^{\gamma_{ca}(K_n)}.$$

Proof: As $\gamma_a(G) = \gamma_{ca}(G) = \lfloor n \div 2 \rfloor + 1$ and every ADS is a $CADS$ also, proof is similar to the above.

Theorem 3.3. For a path P_p with ≥ 5, $D_{CA}(P_p, x) = x^p + 2x^{p-1} + x^{p-2}$.

Proof: Let P_p be a path with the vertex set $\{v_1, v_2, v_3, \ldots\ldots v_{p-1}, v_p\}$. We know

$\gamma_{ca}(P_p) = p - 2$. The only $CADS$ with $(p-2)$ vertices is $\{v_2, v_3, \ldots\ldots v_{p-1}\}$. The only two $CADS$ of cardinality $(p-1)$ are $\{v_1, v_2, v_3, \ldots\ldots v_{p-1}\}$ and $\{v_2, v_3, \ldots\ldots v_{p-1}, v_p\}$. Further the only $CADS$ of cardinality p is the vertex set itself. Thus,

$$D_{CA}(P_p, x) = x^p + 2x^{p-1} + x^{p-2}.$$

Theorem 3.4. For a cycle C_n, $CADP$ is $D_{CA}(C_n, x) = x^{n-2} + nx^{n-1} + nx^{n-2}$.

Proof: As every connected dominating set is a $CADS$ also, result is valid.

Theorem 3.5. ADP of $K_{m,n}$ is

1) $D_A(K_{m,n}, x) = x^{2n} + 2nC_{2n-1}x^{2n-1} + 2nC_{2n-2}x^{2n-2} + \cdots + 2nC_{n+1}x^{n+1}$ $if\ m = n.$

2) $D_A(K_{m,n}, x) = x^m + (m+n)C_{m+1}x^{m+1} + (m+n)C_{m+2}x^{m+2} + \cdots + x^{m+n}$ $if\ m < n\ and\ n = m + 1.$

3) $D_A(K_{m,n}, x) = x^m + \{nC_1x^{m+1} + nC_2x^{m+2} + nC_3x^{m+3} + \cdots + nC_{p-1}x^{m+p-1}\} + (m+n)C_{m+p}x^{m+p} + \cdots + x^{m+n}$

$if m < n, n = m + p\ and\ p > 1.$

Proof: We consider the following 3 cases:

1) If $m = n$, then $\gamma_a(K_{m,n}) = m + 1$. Thus no vertex set with $< (m+1)$ vertices will be an ADS. We can select $2nC_{n+1}$ ADS with $(n+1)$ vertices, $2nC_{2n-1}$ ADS with $(2n-1)$ vertices, $2nC_{2n-2}$ ADS with $(2n-2)$ vertices and so on. Thus,

$$D_A(K_{m,n}, x) = x^{2n} + 2nC_{2n-1}x^{2n-1} + 2nC_{2n-2}x^{2n-2} + \cdots + 2nC_{n+1}x^{n+1}.$$

2) If $m < n$, then $\gamma_a(K_{m,n}) = m$. Here we consider $n = m + 1$. We can select m vertices and one out of n vertices to have ADS of cardinality $(m+1)$, We

can select m vertices and two out of n vertices to have ADS of cardinality $(m+2)$ and so on. Thus

$$D_A(K_{m,n}, x) = x^m + (m+n)C_{m+1}x^{m+1} + (m+n)C_{m+2}x^{m+2} + \cdots + x^{m+n}$$

3) If $m < n$, then $\gamma_a(K_{m,n}) = m$. Here we consider $n = m + p$ where $p > 1$. We can select m vertices and one out of n vertices to have ADS of cardinality $(m+1)$, We can select m vertices and two out of n vertices to have ADS of cardinality $(m+2)$ and so on. Further we can select $n = m + p$ number of ADS in $(m+n)C_{m+p}$ ways and so on. Thus,

$$D_A(K_{m,n}, x) = x^m + \{nC_1x^{m+1} + nC_2x^{m+2} + nC_3x^{m+3} + \cdots + nC_{p-1}x^{m+p-1}\} + (m+n)C_{m+p}x^{m+p} + \cdots + x^{m+n}$$

Corollary 3.6. For the star $K_{1,2}$, $D_A(K_{1,2}, x) = x + 3x^2 + x^3$.

Corollary 3.7. For a star $K_{1,n}$, where $n \geq 3$, and $n = 1 + p$,

$$D_A(K_{1,n}, x) = x + nC_1x^2 + nC_2x^3 + \cdots + nC_{p-1}x^p + (n+1)C_{p+1}x^{p+1} + \cdots + x^{n+1}.$$

Theorem 3.8. $CADP$ of $K_{m,n}$, $2 \leq m \leq n$, is

$$D_{CA}(K_{m,n}, x) = nC_1x^{m+1} + nC_2x^{m+2} + nC_3x^{m+3} + \cdots + x^{m+n}$$

As $\gamma_{ca}(K_{m,n}) = m + 1$, minimum cardinality of a $CADS$ is $(m+1)$. To get this $CADS$, we need to include all m vertices and one out of n vertices. Similarly to get a $CADS$ of cardinality $(m+2)$, we need to include all m vertices and two out of n vertices and so on. The only $CADS$ of cardinality $(m+n)$ is the vertex set itself. Thus $D_{CA}(K_{m,n}, x) = \binom{n}{1}_1 x^{m+1} + nC_2x^{m+2} + nC_3x^{m+3} + \cdots + x^{m+n}$.

Corollary 3.9. For a star $K_{1,n}$, where ≥ 3,

$$D_{CA}(K_{1,n}, x) = nC_1x^2 + nC_2x^3 + \cdots + nC_{n-1}x^n + x^{1+n}.$$

Lemma 3.10. For any bi-star $B(m,n)$, $\gamma_a(B(m,n)) = 2$.

Theorem 3.11. For any bi- star $B(m,n)$ $D_A(B(m,n), x) = x^2(1+x)^{m+n}$.

Proof: Consider a bi-star $B(m,n)$ as shown in Figure 2.

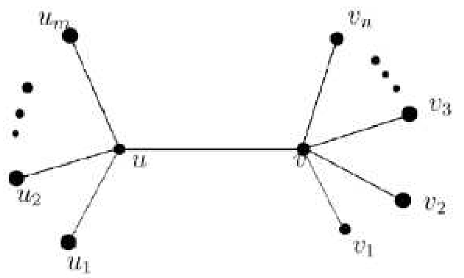

Figure 2

Here $\gamma_a(B(m,n))$ is 2 by Lemma 3.6. Hence the least power of x in accurate domination polynomial is x^2. Any accurate dominating set of cardinality 3 will obviously include the vertices u and v and the third vertex should be selected out on $m+n$ vertices. Thus the bi-star has $(m+n)C_1$ number of accurate dominating sets of cardinality three. Similarly, we can select $(m+n)C_2$ number of accurate dominating sets of cardinality four and so on. Hence the ADP of a bi-star is

$$\begin{aligned} D_a(B(m,n),x) &= x^2 + \binom{m+n}{1}x^3 + \binom{m+n}{2}x^4 + \cdots + x^{m+n+2} \\ &= x^2\left(\binom{m+n}{1}x + \binom{m+n}{2}x^2 + \cdots + x^{m+n}\right) \\ &= x^2(1+x)^{m+n}. \end{aligned}$$

Theorem 3.12. For any bi- star $B(m,n)$ $D_{CA}(B(m,n),x) = x^2(1+x)^{m+n}$.

Proof: As every ADS of a bi-star is a $CADS$ also, result is obvious.

Theorem 3.13. Two isomorphic graphs will have same $ADPs$.

Proof: When two graphs are isomorphic, there will be a one- one correspondence between the vertices and edges of both the graphs and hence the result is obvious.

Remark 3.14. We shall illustrate, with two graphs G_1 and G_2, as shown in the following figures 3 and 4, that the converse of the Theorem 3.9 is not always true.
For both the graphs G_1 and G_2, ADP is $D_A(G_1,x) = D_A(G_2,x) = x^5 + 5x^4 + 10x^3$.
But graphs are not isomorphic as a vertex of degree 4 exists in one graph, not in the other.

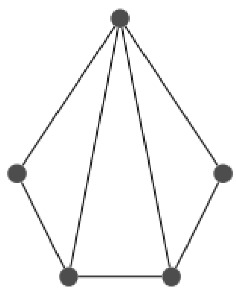

Figure 3: G_1

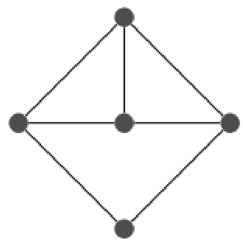

Figure 4: G_2

Theorem 3.15. Let G be a healthy spider graph with $2m+1$ vertices which is constructed by the sub-division of each of the edges present in the star graph $K_{1,m}$ with $m \geq 3$. Then

$$\begin{aligned} D_A(G,x) = x^m + \binom{m+1}{1}x^{m+1} + \binom{m+1}{2}x^{m+2} + \cdots + \binom{m+1}{m-1}x^{2m-1} \\ + \left(1 + \binom{m}{m-1}\right)x^{2m} + x^{2m+1}. \end{aligned}$$

Proof: Consider a healthy spider with $2m+1$ number of vertices as shown in Figure 5.

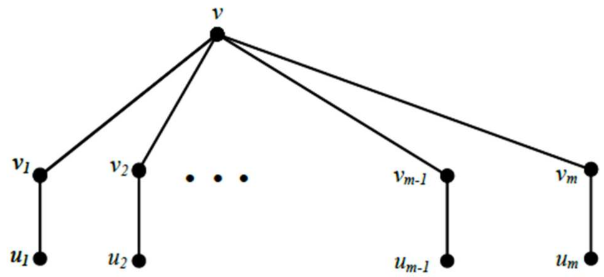

Figure 5: Healthy spider graph

Let $A = \{v_1, v_2, v_3, \dots\dots v_{m-1}, v_m\}$ and $B = \{u_1, u_2, u_3, \dots\dots u_{m-1}, u_m\}$.
We note that the graph has only one ADS namely A of cardinality m. Any ADS of cardinality $(m+1)$ will have all elements of set A and one element selected out of vertex set of $G - A = B + \{v\}$. Similar selection of vertices can get us ADS of cardinalities $(m+2), (m+3) \dots (2m-1)$. ADS of cardinality $2m$ can be obtained in the following two ways:

a) Select all vertices $\{v_1, v_2, v_3, \dots\dots v_{m-1}, v_m, u_1, u_2, u_3, \dots\dots u_{m-1}, u_m\}$ - only one way of selection.
b) Select all vertices of the set A, vertex v and $(m-1)$ vertices selected out of set B. Selection can be done in $\binom{m}{m-1}$ ways.

Obviously, there is only one ADS of cardinality $(2m+1)$. Thus the ADP of a healthy spider G is:

$$D_A(G,x) = x^m + \binom{m+1}{1} x^{m+1} + \binom{m+1}{2} x^{m+2} + \cdots + \binom{m+1}{m-1} x^{2m-1} + \left(1 + \binom{m}{m-1}\right) x^{2m} + x^{2m+1}$$

Theorem 3.16. For a healthy spider G,

$$D_{CA}(G,x) = x^{m+1} + \binom{m}{1} x^{m+2} + \binom{m}{2} x^{m+3} + \cdots + \binom{m}{m-1} x^{2m} + x^{2m+1}$$

Proof: We shall consider the healthy spider as shown in Figure 5. As usual, we shall take $A = \{v_1, v_2, v_3, \dots\dots v_{m-1}, v_m\}$ and $B = \{u_1, u_2, u_3, \dots\dots u_{m-1}, u_m\}$.$CADS$ of minimum cardinality is $(m+1)$ and only such $CADS$ is the set $C = \{v_1, v_2, v_3, \dots\dots v_{m-1}, v_m, v\}$. Any $CADS$ of cardinality $\geq (m+2)$ will have $(m+1)$ vertices of C and remaining vertices taken out of B. Thus,

$$D_{CA}(G,x) = x^{m+1} + \binom{m}{1} x^{m+2} + \binom{m}{2} x^{m+3} + \cdots + \binom{m}{m-1} x^{2m} + x^{2m+1}.$$

Theorem 3.17. Let G be a wounded spider graphs having $m+s+1$ vertices, which is constructed by the sub-division of s out of m edges present in the star graph $K_{1,m}$ with $m \geq 3$. Then

$$D_A(G,x) = x^m + \binom{s+1}{1} x^{m+1} + \binom{s+1}{2} x^{m+2} + \cdots + \binom{s+1}{s-1} x^{m+s-1}$$

$$+\left(1+\binom{s}{s-1}\right)x^{m+s}+x^{m+s+1}$$

Proof: Consider a wounded spider as required in the statement of the theorem as shown in the Figure 6.

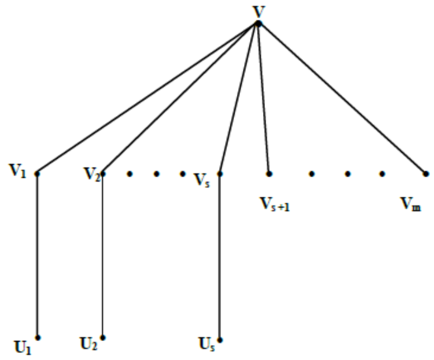

Figure 6: Wounded spider graph

Let $A=\{V_1,V_2,V_3\ldots V_s,V_{s+1},V_{s+2},\ldots V_m\}$, $B=\{U_1,U_2,U_3\ldots U_s\}$.
We note that the graph has only one ADS namely set A of cardinality m. Any ADS of cardinality $(m+1)$ will have all elements of set A and one element selected out of vertex set of $G-A=B+\{v\}$. Similar selection of vertices can get us ADS of cardinalities $(m+2),(m+3)\ldots(m+s-2)$. ADS of cardinality $m+s$ can be obtained in the following two ways:

a) Select all vertices $\{V_1,V_2,V_3\ldots V_s,V_{s+1},V_{s+2},\ldots V_m,U_1,U_2,U_3\ldots U_s\}$ - only one way of selection.
b) Select all vertices of the set A, vertex v and $(s-1)$ vertices selected out of set B. Selection can be done in $\binom{s}{s-1}$ ways.

Obviously, there is only one ADS of cardinality $(m+s+1)$. Thus the ADP of a wounded spider G is:

$$D_A(G,x)=x^m+\binom{s+1}{1}x^{m+1}+\binom{s+1}{2}x^{m+2}+\cdots+\binom{s+1}{s-1}x^{m+s-1}$$
$$+\left(1+\binom{s}{s-1}\right)x^{m+s}+x^{m+s+1}.$$

Theorem 3.18. For a wounded spider G

$$D_{CA}(G,x)=x^{m+1}+\binom{s}{1}x^{m+2}+\binom{s}{2}x^{m+3}+\cdots+\binom{s}{s-1}x^{m+s}+x^{m+s+1}.$$

Proof: We shall consider a wounded spider as shown in Figure 6. As usual, we shall take $A=\{V_1,V_2,V_3\ldots V_s,V_{s+1},V_{s+2},\ldots V_m\}$, $B=\{U_1,U_2,U_3\ldots U_s\}$. $CADS$ of minimum cardinality is $(m+1)$ and only such $CADS$ is the set $C=\{V_1,V_2,V_3\ldots V_s,V_{s+1},V_{s+2},\ldots V_m,V\}$. Any $CADS$ of cardinality $\geq(m+2)$ will have $(m+1)$ vertices of C and remaining vertices taken out of B. Thus

$$D_{CA}(G,x)=x^{m+1}+\binom{s}{1}x^{m+2}+\binom{s}{2}x^{m+3}+\cdots+\binom{s}{s-1}x^{m+s}+x^{m+s+1}.$$

Theorem 3.19. For a connected graph of order n

$i)\ d_A\,(G,n) = 1$
$ii)\ d_A\,(G,i) = 0\ if\ i < \ \gamma_a(G)$ or $i > n.$
$iii)\ D_A(G,x)$ will not have a constant term. That is zero is a root of ADP.

Theorem 3.20. For a connected graph of order n
$i)\ d_{CA}\,(G,n) = 1$
$ii)\ d_{CA}\,(G,i) = 0\ if\ i < \ \gamma_{ca}(G)$ or $i > n.$
$iii)\ D_{CA}(G,x)$ will not have a constant term. That is zero is a root of ADP.

4. Conclusion

ADP and $CADP$ represent two new way in which we can represent a graph algebraically. We can find ADP and $CADP$ of more standard graphs. We can also compare these two new polynomials obtained with other existing domination polynomials of the graph.

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Conflict of interest. The authors declare that they have no conflict of interest.

Authors' Contributions. All the authors contributed equally to this work.

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