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More on Fuzzy Pre-γ-Open and Fuzzy Pre-γ-generalized Closed Sets

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Abstract. This study aims to continue the study of the properties of pre-γ-open and pre-γ-generalized closed sets in fuzzy topological spaces. Also, we introduce the concepts of fuzzy pre-γ-closed, fuzzy pre-γ-closure, fuzzy pre-γ-interior, and fuzzy pre-γ-generalized open sets. We prove that every fuzzy pre-γ-closed set is fuzzy pre-γ-generalized-closed but not converse. In addition, we introduce some characterizations and properties of these concepts. Finally, we investigate the relationship between these fuzzy sets.

Keywords: Fuzzy pre-γ-open, fuzzy pre-γ-closed, fuzzy pre-γ-closure, fuzzy pre-γ-interior, fuzzy pre-γ-generalized closed.

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1. Introduction

The idea of fuzzy sets originated from the classical paper of Zadeh [11] in 1965. The theory of fuzzy topological spaces was introduced and developed by Chang [4] in 1968 and since then many concepts in general topology have been extended to fuzzy topological spaces. In general topology, the concept of pre-γ-open sets was introduced and studied also by Ibrahim [6]. Properties of pre-γ-open sets and mappings are also discussed by Vadivel and Sivashanmugaraja [10]. Generalized maximal closed set in a topological space is studied by Banasode and Desurkar [2]. In 1979, Kasahara [8] defined the notion of an operation γ on fuzzy topological spaces. Kalitha and Das [7] introduced the notion of fuzzy γ-open sets. Fuzzy generalized γ-closed sets are introduced by De [5]. In [9], the concept of the pre-γ-open set has been generalized to the fuzzy setting. In this paper fuzzy pre-γ-closed, fuzzy pre-γ-closure, fuzzy pre-γ-interior and fuzzy pre-γ-generalized open sets are introduced. Notations, definitions and preliminaries appear in section 2. The main results of the paper are given in sections 3 and 4. In section 3, the properties of pre-γ-open fuzzy sets are discussed. In section 4, we introduce the notion of pre-γ-generalized open sets and investigate the relationships between these fuzzy sets.

2. Preliminaries

Throughout this paper (X,τ) or simply X stand for a fuzzy topological space (fts, for short). The interior, the closure and complement of a fuzzy set $A \in I^X$ is denoted by int(A), cl(A) and A^c respectively. By 0_X and 1_X, we mean the constant fuzzy sets taking on the values 0 and 1 on X, respectively. Now we recall some of the fundamental definitions in fuzzy topology.

Definition 2.1. [7] A fuzzy operation $\gamma : \tau \rightarrow I^X$ such that $\mu \subseteq \gamma(\mu)$, for every $\mu \in \tau$, where $\gamma(\mu)$ denotes the value of γ at μ. The mapping definedas $\gamma(\mu)=\mu$, $\gamma(\mu)=cl(\mu)$, $\gamma(\mu)=int(cl(\mu))$, etc are examples of fuzzy operations.

Definition 2.2. [7] A fuzzy subset λ of a fts (X,τ) is called a fuzzy γ-open, if $\forall\ p_x^\lambda q\ \lambda$, $\exists\ \mu \in \tau$ and $p_x^\lambda q\ \mu$ such that $\gamma(\mu) \subseteq \lambda$. τ_γ denotes the set of all fuzzy γ-open sets. Clearly we have $\tau_\gamma \subseteq \tau$.

Definition 2.3. [7] Let λ be a fuzzy set in a fts X. Then τ_γ-cl(λ) is defined as $\tau_\gamma\text{-cl}(\lambda) = \wedge\{\mu : \lambda \leq \mu,\ \mu^c \in \tau_\gamma(X)\}$ and τ_γ-int(λ) is defined as $\tau_\gamma\text{-int}(\lambda) = \vee\{\mu : \mu \leq \lambda,\ \mu \in \tau_\gamma(X)\}$.

Definition 2.4. [3] A fuzzy subset μ of a fts (X,τ) is called fuzzy preopen if $\mu \leq int(cl(\mu))$.

Definition 2.5. [3] Let μ be a fuzzy subset of (X,τ). Then the fuzzy pre- interior of μ is defined by $pint(\mu) = \vee\{\lambda \leq \mu : \lambda \in FPO(X)\}$ and fuzzy pre-closure of μ is defined by $pcl(\mu) = \wedge\{\lambda \geq \mu : \lambda \in FPC(X)\}$.

Definition 2.6. [9] A fuzzy subset μ of (X,τ) is said to be fuzzy pre-γ-open (in short, fp_γ-open) if $\mu \leq \tau_\gamma\text{-int}(cl(\mu))$. The family of all pre-$\gamma$-open fuzzy sets is denoted by $FP_\gamma O(X)$.

Definition 7.2.17. [1] A fts (X,τ) is called fuzzy door space if each fuzzy subset of X is fuzzy open or fuzzy closed.

3. Fuzzy pre-γ-open and pre-γ-closed sets

In this section, we introduce the concepts of pre-γ-closed, pre-γ-closure and pre-γ-interior in fuzzy settings. Also we investigate somecharacterizations and fundamental properties of pre-γ-closed and pre-γ-open fuzzy sets.

Definition 3.1. A fuzzy subset μ of (X,τ) is called fuzzy pre-γ-closed (in short, fp_γ-closed) if $\mu \geq \tau_\gamma\text{-cl}(int(\mu))$. The family of all fuzzy pre-γ-closed sets is represented by $FP_\gamma C(X)$.

Theorem 3.1. Let μ be any fuzzy subset of a fts X,

(i) μ is fp_γ-open set iff μ^c is fp_γ-closed;

(ii) μ is fp_γ-closed set iff μ^c is fp_γ-open.

Definition 3.2. Let μ be a fuzzy subset of a (X,τ). Then

(i) fuzzy pre-γ-interior of μ is defined by $pint_\gamma(\mu) = \vee\{\lambda \leq \mu : \lambda \in FP_\gamma O(X)\}$.

(ii) fuzzy pre-γ-closure of μ is defined by $pcl_\gamma(\mu) = \wedge\{\lambda \geq \mu : \lambda \in FP_\gamma C(X)\}$.

Theorem 3.2. For any fuzzy set μ of a fts X, the following statements hold
(i) $int_\gamma(\mu) \leq int(\mu) \leq pint(\mu) \leq \mu \leq pcl(\mu) \leq cl(\mu) \leq cl_\gamma(\mu)$;
(ii) $int_\gamma(\mu) \leq pint_\gamma(\mu) \leq pint(\mu) \leq \mu \leq pcl(\mu) \leq pcl_\gamma(\mu) \leq cl_\gamma(\mu)$;

Example 3.1. Let X = {a, b, c} and $\mu_1, \mu_2, \mu_3, \mu_4 \in I^X$ defined by $\mu_1(a)=0.3$, $\mu_1(b)=0.4$, $\mu_1(c)=0.7$; $\mu_2(a)=0.5$, $\mu_2(b)=0.5$, $\mu_2(c)=0.5$; $\mu_3(a)=0.5$, $\mu_3(b)=0.5$, $\mu_3(c)=0.7$; $\mu_4(a)=0.3$, $\mu_4(b)=0.4$, $\mu_4(c)=0.5$. Let $\tau = \{\underline{0}, \underline{1}, \mu_1, \mu_2, \mu_3, \mu_4\}$. Now clearly (X,τ) is a fts. Define $\gamma: \tau \to I^X$ by $\gamma(\underline{1})=\underline{1}$, $\gamma(\underline{0})=\underline{0}$, $\gamma(\mu_1)=\mu_1$, $\gamma(\mu_2)=\mu_2$, $\gamma(\mu_3)=int(cl(\mu_3))$, $\gamma(\mu_4)=cl(\mu_4)$. The fuzzy sets μ_3 and μ_4 are fuzzy preopen and also fuzzy pre-γ-open sets, but not fuzzy γ-open.

Theorem 3.3. Let μ_1 and μ_2 be two fuzzy sets of a fts (X, τ). Then
(i) $\mu_1 \leq \mu_2$ iff $pint_\gamma(\mu_1) \leq pint_\gamma(\mu_2)$;
(ii) $\mu_1 \leq \mu_2$ iff $pcl_\gamma(\mu_1) \leq pcl_\gamma(\mu_2)$.
Proof: Obvious.

Theorem 3.4. For any fuzzy subset μ of a fts X, the following statementshold
(i) $pcl_\gamma(\mu^c) = (pint_\gamma(\mu))^c$;
(ii) $pint_\gamma(\mu^c) = (pcl_\gamma(\mu))^c$.
Proof:

(i) $(pint_\gamma(\mu))^c = (\vee\{d: d \leq \mu, d \in FP_\gamma O(X)\})^c$
$= \wedge\{d^c : d \leq \mu, d \in FP_\gamma O(X)\}$
$= \wedge\{c : c \geq \mu^c, c \in FP_\gamma C(X)\}$
$= pcl_\gamma(\mu^c)$.

(ii) $(pcl_\gamma(\mu))^c = (pcl_\gamma(\mu^c)^c)^c$
$= ((pint_\gamma(\mu^c))^c)^c$
$= pint_\gamma(\mu^c)$

Theorem 3.5. For any fuzzy subset μ of a fts X, the below statementshold.
(i) μ is fuzzy pre-γ-closed iff $\mu = pcl_\gamma(\mu)$;
(ii) μ is fuzzy pre-γ-open iff $\mu = pint_\gamma(\mu)$.
Proof: (i) Suppose $\mu = pcl_\gamma(\mu) = \wedge\{\lambda$: λ is a pre-γ-closed fuzzy set and $\lambda \geq \mu\}$ that implies, $\mu \in \wedge\{\lambda$: λ is a pre-γ-closed fuzzy set and $\lambda \geq \mu$ that implies, μ is pre-γ-closed fuzzy set.

Conversely, suppose μ is a pre-γ-closed fuzzy set in X. We take $\mu \leq \mu$ and μ is a fuzzy pre-γ-closed fuzzy set. So $\mu \in \{\lambda$: λ is a pre-γ-closed fuzzy set and $\lambda \geq \mu\}$, $\mu \leq \lambda$ implies, $\mu = \wedge\{\lambda$: λ is a pre-γ-closed fuzzy set and $\lambda \geq \mu\} = pcl_\gamma(\mu)$.
(ii) Similar to that of (i).

Theorem 3.6. In a fts X, the following statements hold for fuzzy pre-γ-closure.
(i) $pcl_\gamma(0_X) = 0_X$;
(ii) $pcl_\gamma(1_X) = 1_X$;

(iii) $pcl_\gamma(\mu)$ is a pre-γ-closed fuzzy set in X;
(iv) $pcl_\gamma(pcl_\gamma(\mu)) = pcl_\gamma(\mu)$.

Theorem 3.7. For any two fuzzy subsets μ_1 and μ_2 of a fts X, the belowstatements hold.

(i) $pcl_\gamma(\mu_1 \vee \mu_2) \geq pcl_\gamma(\mu_1) \vee pcl_\gamma(\mu_2)$;
(ii) $pcl_\gamma(\mu_1 \wedge \mu_2) \leq pcl_\gamma(\mu_1) \wedge pcl_\gamma(\mu_2)$;

Proof: (i) Since $\mu_1 \leq \mu_1 \vee \mu_2$ (or) $\mu_2 \leq \mu_1 \vee \mu_2$, which gives $pcl_\gamma(\mu_1) \leq pcl_\gamma(\mu_1 \vee \mu_2)$ (or) $pcl_\gamma(\mu_2) \leq pcl_\gamma(\mu_1 \vee \mu_2)$. Therefore $pcl_\gamma(\mu_1 \vee \mu_2) \geq pcl_\gamma(\mu_1) \vee pcl_\gamma(\mu_2)$.
(ii) Similar to that of (i).

Theorem 3.8. In a fts X, the following statements hold for fuzzy pre-γ-interior.
(i) $pint_\gamma(0_X) = 0_X$;
(ii) $pint_\gamma(1_X) = 1_X$;
(iii) $pint_\gamma(\mu)$ is a pre-γ-open fuzzy set in X.
(iv) $pint_\gamma(pint_\gamma(\mu)) = pint_\gamma(\mu)$.

Theorem 3.9. For any two fuzzy subsets μ_1 and μ_2 of a fts X, the following statements hold.
(i) $pint_\gamma(\mu_1 \vee \mu_2) \geq pint_\gamma(\mu_1) \vee pint_\gamma(\mu_2)$;
(ii) $pint_\gamma(\mu_1 \wedge \mu_2) \leq pint_\gamma(\mu_1) \wedge pint_\gamma(\mu_2)$.

Proof: (i) Since $\mu_1 \leq \mu_1 \vee \mu_2$ (or) $\mu_2 \leq \mu_1 \vee \mu_2$ that implies $pint_\gamma(\mu_1) \leq pint_\gamma(\mu_1 \vee \mu_2)$ (or) $pint_\gamma(\mu_2) \leq pint_\gamma(\mu_1 \vee \mu_2)$. Therefore, $pint_\gamma(\mu_1 \vee \mu_2) \geq pint_\gamma(\mu_1) \vee pint_\gamma(\mu_2)$.
(ii) Similar to that of (i).

Proposition 3.1. Let (X,τ) be fuzzy γ-regular and fuzzy door space. Then each pre-γ-open fuzzy set is an open fuzzy set.

4. Fuzzy pre-γ-generalized open and pre-γ-generalized closed sets

In this section, we introduce the concept of pre-γ-generalized open sets in fuzzy topological spaces. Further we discuss the relationships between fuzzy pre-γ-open and fuzzy pre-γ-generalized open sets.

Definition 4.1. Let γ be a fuzzy operation on a fts (X,τ). A fuzzy subset $\mu \in I^X$ is called fuzzy pre-γ-generalized open (in short, $fp_\gamma g$-open) if $\mu \leq pint_\gamma(\mu)$ whenever $\mu \leq \lambda$ and λ is pre-γ-closed fuzzy set in (X,τ).

Remark 4.1. For any fuzzy subset μ in a fuzzy topological space X,
(i) μ is $fp_\gamma g$-open iff μ^c is $fp_\gamma g$-closed;
(ii) μ is $fp_\gamma g$-closed iff μ^c is $fp_\gamma g$-open.

Theorem 4.1. A fuzzy subset μ of a fts (X,τ) is $fp_\gamma g$-open iff $\lambda \leq pint_\gamma(\mu)$, whenever λ is pre-γ-closed fuzzy set and $\lambda \leq \mu$.

Proof: Let μ be a $fp_\gamma g$-open in X. Then μ^c is $fp_\gamma g$-closed in X. Let λ be a fp_γ-closed in X such that $\lambda \leq \mu$. So $\mu^c \leq \lambda^c$, λ^c is fp_γ-open in X. Since μ^c is $fp_\gamma g$ -closed, $pcl_\gamma(\mu^c) \leq \lambda^c$, which gives $(pint_\gamma(\mu))^c \leq \lambda^c$. Hence $\lambda \leq pint_\gamma(\mu)$.

Conversely, suppose that $\lambda \leq pint_\gamma(\mu)$, whenever $\lambda \leq \mu$ and λ is fp_γ-closed in X . So $(pint_\gamma(\mu))^c \leq \lambda^c = \nu$, where ν is pre-γ-open fuzzy set fp_γ-open in X. That is $pcl_\gamma(\mu^c) \leq \nu$, which gives μ^c is $fp_\gamma g$ -closed. Thus μ is $fp_\gamma g$-open.

Theorem 4.2. Every closed fuzzy set in (X,τ) is $fp_\gamma g$-closed in (X,τ).

Proof: Let μ be a closed fuzzy set in fts X. Let $\mu \leq \lambda$, where λ is pre-γ- open fuzzy set in X. Since μ is fuzzy closed, it is also is pre-γ-closed and so $cl_\gamma(\mu) = pcl_\gamma(\mu) = \mu \leq \lambda$. Thus $pcl_\gamma(\mu) \leq \lambda$. Hence μ is $fp_\gamma g$-closed.

Theorem 4.3. Every fuzzy γ-closed set in (X,τ) is $fp_\gamma g$-closed in (X,τ).

Proof: Let μ be a γ-closed fuzzy set in fts X. Let $\mu \leq \lambda$, where λ is γ-open fuzzy set in X. Since μ is γ-closed, it is also is pre-γ-closed and so $cl_\gamma(\mu) = pcl_\gamma(\mu) = \mu \leq \lambda$. Thus $pcl_\gamma(\mu) \leq \lambda$. Thus μ is $fp_\gamma g$ -closed.

Theorem 4.4. Every pre-γ-closed fuzzy set in (X,τ) is $fp_\gamma g$-closed in (X, τ).

Proof: Let μ be a pre-γ-closed fuzzy set in fts X. Let $\mu \leq \lambda$, where λ is pre-γ-open fuzzy set in X. By hypothesis, μ is pre-γ-closed fuzzy set, we have $pcl_\gamma(\mu) = \mu \leq \lambda$. Thus $pcl_\gamma(\mu) \leq \lambda$. Hence μ is $fp_\gamma g$ -closed.

But the converse need not be true as shown in the following example.

Example 4.1. Let $X = \{a,b\}$. Consider the fuzzy sets $\mu_1, \mu_2, \in I^X$ defined by $\mu_1(a)=0.2$, $\mu_1(b)=0.4$; $\mu_2(a)=0.3$, $\mu_2(b)=0.6$. Let $\tau = \{\underline{1},\underline{0},\mu_1\}$. Now clearly (X,τ) is a fts. Define γ: $\tau \rightarrow I^X$ by $\gamma(\underline{1})=\underline{1}$, $\gamma(\underline{0})=\underline{0}$, $\gamma(\mu_1)=\mu_1$. The fuzzy set μ_2 is $fp_\gamma g$-closed set, but not fp_γ -closed set.

Theorem 4.5. If μ is pre-γ-generalized closed and pre-γ-open fuzzy set in X, then μ is fuzzy pre-γ-closed.

Proof: Let $\mu \leq \mu$, where μ is pre-γ-generalized closed and pre-γ-open fuzzy set. So $pcl_\gamma(\mu) \leq \mu$. But $\mu \leq pcl_\gamma(\mu)$. Thus, $\mu = pcl_\gamma(\mu)$ and so μ is pre-γ-closed fuzzy set.

Theorem 4.6. Let μ be a fuzzy subset of a fts (X,τ). Then μ is a pre-γ-generalized-closed fuzzy set iff $\mu \leq \lambda$ implies $pcl_\gamma(\mu) \leq \lambda$, $\forall$ pre-γ-closed fuzzy set λ in X.

Proof: Let μ be a pre-γ-generalized-closed fuzzy set and λ be a pre-γ-closed fuzzy set such that $\mu \leq \lambda$. Thus $\mu \leq \lambda^c$ and λ^c is pre-γ-open fuzzyset in X. Since μ is pre-γ-generalized-closed fuzzy set, $pcl_\gamma(\mu) \leq \lambda^c$. Thus $pcl_\gamma(\mu) = \lambda$.

Conversely, let ν be pre-γ-open fuzzy set in X, such that $\mu \leq \lambda$. Thus $\mu \bar{q} \nu^c$ which implies $pcl_\gamma(\mu) \leq \nu^c$, so $pcl_\gamma(\mu) \leq \nu$. Thus μ is pre-γ-generalized-closed fuzzy set in X.

Theorem 4.7. If μ is pre-γ-generalized-closed fuzzy set in a fts (X,τ) and $\mu \leq \lambda \leq pcl_\gamma(\mu)$, then λ is also a pre-γ-generalized-closed fuzzy set in (X,τ).

Proof: Let ν be pre-γ-open fuzzy set in X such that $\lambda \leq \nu$. So $\mu \leq \lambda, \mu \leq \nu$. By hypothesis μ is a pre-γ-generalized-closed fuzzy set in X, it follows that $pcl_\gamma(\mu) \leq \nu$. Now

$\lambda \leq pcl_\gamma(\mu)$ which gives $pcl_\gamma(\lambda) \leq pcl_\gamma(pcl_\gamma(\mu)) = pcl_\gamma(\mu)$. Hence $pcl_\gamma(\lambda) \leq \nu$. Thus λ is fuzzy pre-γ-generalized-closed in X.

Theorem 4.8. If μ is pre-γ-generalized-open fuzzy set in a fts (X,τ) and $pint_\gamma(\mu) \leq \lambda \leq \mu$, then λ is also a pre-γ-generalized-open fuzzy set in (X,τ).

Proof: Let μ be fuzzy pre-γ-generalized-open and λ be any fuzzy set in X such that $pint_\gamma(\mu) \leq \lambda \leq \mu$. Then μ^c is a pre-γ-generalized-closed fuzzy set in X and $\mu^c \leq \lambda^c \leq pcl_\gamma(\mu^c)$. Then, λ^c is a pre-γ-generalized-closed fuzzy set in X. Hence λ is fuzzy pre-γ-generalized-open set in X.

Theorem 4.9. Let μ be any fuzzy subset of (Y,τ_Y) and (Y,τ_Y) be a subspace of a fts (X, τ_X). If μ is $fp_\gamma g$-closed set in X, then μ is $fp_\gamma g$-closed in Y.

5. Conclusion

In this paper, we introduced the concepts of fuzzy pre-γ-closed, fuzzy pre-γ-closure, fuzzy pre-γ-interior and fuzzy pre-γ-generalized open sets in fuzzy topological spaces. We discussed the relationships between fuzzy pre-γ-closed and fuzzy pre-γ-generalized closed sets. We proved that every pre-γ-closed fuzzy set is pre-γ-generalized-closed as well as every γ-closed fuzzy set is pre-γ-generalized-closed but not converse. There is a scope to study and extend these newly defined generalized fuzzy sets.

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