Short communication
An Integral Involving a Generalized Hypergeometric Function

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Abstract. In 1961, MacRobert in his very popular, useful and interesting research paper obtained a new type of finite integrals and used the integrals to evaluate integral involving E-functions which he had developed and is a generalization of hypergeometric and generalized hypergeometric functions. The main objective of this short research paper is to find an exciting integral associated with a generalized hypergeometric function by using the integrals obtained by MacRobert. The beauty of our results is that they appear on the product of the ratios of gamma functions. It is clear that the integral associated with gamma functions, the results are very useful from the perspective of the point of view of applications. In terms of parameters, one can easily derive the known integrals due to Rathie and the integral given in Mathai and Saxena’s book. It is no exaggeration to mention here that, for other integrals, the transformation and summation formulas involve generalized hypergeometric function.

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1. Introduction
Fernandez et al., [1] discussed the following bivariate Mittage-Leffler function that occurs naturally in biological engineering, its fractional calculus properties, and its associations with other special functions such as bivariate Laguerre polynomials:

\[ E_{\delta, \eta, \rho}^\tau (w, y) = \sum_{r,s=0}^{\infty} \frac{(\tau)_r (\rho)_s}{\Gamma(\delta r + \eta s + \rho)} \frac{w^r y^s}{r! s!}, \]  

(1)

where \( \delta, \eta, \rho, \tau, w, y \in \mathbb{C} \) with \( \text{Re}(\delta) > 0, \text{Re}(\eta) > 0 \).
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As recently as 2021, Abubakar and Dudi [2] used the two-variable Mittage-Leffler function in equation (1) as a kernel to study some properties and applications to statistical and fractional calculus of the following new generalized beta function:

$$B_{\delta,\lambda,\rho}^{\tau,p,q}(x,z) = \int_{0}^{1} t^{\delta-1} (1-t)^{\lambda-1} E_{\delta,\lambda,\rho}^{\tau}(\frac{-p}{z},-\frac{q}{1-t}) \, dt,$$

(2)

where \( Re(\delta) > 0, Re(\lambda) > 0, Re(\rho) > 0, Re(\tau) > 0 \).

They also [2] studied the following generalized Gauss and confluent hypergeometric functions defined by the beta function in equation (2):

$$2F_{1;\delta,\lambda,\rho}^{\tau,p,q}(a,b;c;z) = \sum_{r=0}^{\infty} \frac{B_{\delta,\lambda,\rho}^{\tau,p,q}(a+r,b,c-b)}{B(b,c-b)} \frac{z^r}{r!},$$

(3)

where \( p, q \geq 0, |z| < 1, Re(\delta) > 0, Re(\lambda) > 0, Re(\rho) > 0, Re(\tau) > 0, Re(c) > Re(b) > 0, Re(a) > 0 \), and

$$1F_{1;\delta,\lambda,\rho}^{\tau,p,q}(b;c;z) = \sum_{r=0}^{\infty} \frac{B_{\delta,\lambda,\rho}^{\tau,p,q}(b+r,c-b)}{B(b,c-b)} \frac{z^r}{r!},$$

(4)

where \( p, q \geq 0, |z| < 1, Re(\delta) > 0, Re(\lambda) > 0, Re(\rho) > 0, Re(\tau) > 0, Re(c) > Re(b) > 0 \).

2. Results required

The following interesting integral formulas by MacRobert’s [3] will be present in our current investigation:

$$2 \int_{0}^{\pi} e^{\lambda (a+b) \phi} (\sin \phi)^{a-1} (\cos \phi)^{b-1} d\phi = e^{\lambda \frac{\pi}{2} \frac{r(a)r(\beta)}{r(a+b)}},$$

(5)

where \( Re(\alpha) > 0 \) and \( Re(\beta) > 0 \),

$$2 \int_{0}^{\pi} \cos[(a + \beta) \phi] (\sin \phi)^{a-1} (\cos \phi)^{b-1} d\phi = \cos \left( \pi \frac{\alpha}{2} \frac{r(a)r(\beta)}{r(a+b)} \right),$$

(6)

where \( Re(\alpha) > 0 \) and \( Re(\beta) > 0 \), and

$$2 \int_{0}^{\pi} \sin[(a + \beta) \phi] (\sin \phi)^{a-1} (\cos \phi)^{b-1} d\phi = \sin \left( \pi \frac{\alpha}{2} \frac{r(a)r(\beta)}{r(a+b)} \right),$$

(7)

where \( Re(\alpha) > -1 \) and \( Re(\beta) > 0 \).

This paper is motivated by the work of several researchers (see for example, [4]-[10]) who have studied many integrals formulas and transforms involving a wide range of special functions of mathematical physics play an important role in many field such as engineering, science and technology.

3. Main result

The following integral involving a generalized hypergeometric function will be evaluated in this article:

$$2 \int_{0}^{\pi} e^{\lambda (a+b) \phi} (\sin \phi)^{a-1} (\cos \phi)^{b-1} 2F_{1;\delta,\lambda,\rho}^{\tau,p,q}(a,b;c;e^{i\frac{\pi}{2} \sin \phi}) d\phi$$

$$= e^{\lambda \frac{\pi}{2} \frac{r(a)r(\beta)}{r(a+b)}} 2F_{2;\delta,\lambda,\rho}^{\tau,p,q}(a,b,a;\alpha + \beta; 1),$$

(8)

where \( Re(\alpha) > 0 \) and \( Re(\beta) > 0 \).

Proof: Expressing the left-hand side of (8) by \( I \), we have
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\[ I = \int_0^\pi e^{i(\alpha + \beta)\phi} (\sin \phi)^{\alpha - 1} (\cos \phi)^{\beta - 1} \sum_{r=0}^{\infty} \frac{B_{\delta, \rho}(b + r, c - b)}{B(b, c - b)} e^{-i\phi}\frac{1}{2} (\sin \phi)^n d\phi. \]

Changing the order of integration and summation (which is reasonably easy seen to be justified given the uniform convergence of the series involved in the process), after a little simplification, gives

\[ I = \sum_{r=0}^{\infty} \left( a \right)_r \frac{B_{\delta, \rho}(b + r, c - b)}{B(b, c - b)} e^{-i\phi}\frac{1}{2} \int_0^\pi e^{i(\alpha + \beta + n)\phi} (\sin \phi)^{\alpha - 1} (\cos \phi)^{\beta - 1} d\phi. \]

(9)

Evaluating the integral using MacRobert’s (5), we get

\[ I = \sum_{r=0}^{\infty} \left( a \right)_r \frac{B_{\delta, \rho}(b + r, c - b)}{B(b, c - b)} e^{-i\phi}\frac{1}{2} \frac{\Gamma(\alpha + n) \Gamma(\beta)}{\Gamma(\alpha + n + 1)}. \]

And after some simplification, yields

\[ I = e^{\alpha \frac{\pi}{2} \Gamma(\alpha) \Gamma(\beta)} \sum_{r=0}^{\infty} \frac{(\alpha)_r (\alpha)_n B_{\delta, \rho}(b + r, c - b)}{\Gamma(\alpha + \beta) \Gamma(\alpha + n + 1)} \frac{1}{r!}. \]

Finally summing up the series, we easily go to the right-hand side of the (8). Consider equations (6) and (7), which leads to the following result:

**Corollary 3.1.**

\[ \int_0^\pi \cos[(\alpha + \beta)\phi] (\sin \phi)^{\alpha - 1} (\cos \phi)^{\beta - 1} z^{F_{1, \delta, \rho}}(a, b; c; e^{i(\phi - \frac{\pi}{2})\sin \phi}) d\phi \]

\[ = \cos \left( \pi \frac{\alpha}{2} \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \right) z^{F_{1, \delta, \rho}}(a, b; c; a + \alpha + 1), \]

(10)

where \( \text{Re}(\alpha) > 0 \) and \( \text{Re}(\beta) > 0 \), and

\[ \int_0^\pi \sin[(\alpha + \beta)\phi] (\sin \phi)^{\alpha - 1} (\cos \phi)^{\beta - 1} z^{F_{1, \delta, \rho}}(a, b; c; e^{i(\phi - \frac{\pi}{2})\sin \phi}) d\phi \]

\[ = \sin \left( \pi \frac{\alpha}{2} \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \right) z^{F_{1, \delta, \rho}}(a, b; c; a + \alpha + 1), \]

(11)

where \( \text{Re}(\alpha) > -1 \) and \( \text{Re}(\beta) > 0 \).

**5. Special cases**

In this section we will cover some of the very interesting known special cases of our main result.

1. If we take \( \alpha = \alpha + \beta \) in equation (8) we get

\[ \int_0^\pi e^{i(\alpha + \beta)\phi} (\sin \phi)^{\alpha - 1} (\cos \phi)^{\beta - 1} z^{F_{1, \delta, \rho}}(a, b; c; e^{i(\phi - \frac{\pi}{2})\sin \phi}) d\phi \]

\[ = e^{i\phi}\frac{\pi}{2} \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \frac{1}{z^{F_{1, \delta, \rho}}(a; a + \alpha + 1)}. \]

2. If \( \delta = \rho = \tau = 1 \) in equation (8), gives

\[ \int_0^\pi e^{i(\alpha + \beta)\phi} (\sin \phi)^{\alpha - 1} (\cos \phi)^{\beta - 1} z^{F_{1, \delta, \rho}}(a, b; c; e^{i(\phi - \frac{\pi}{2})\sin \phi}) d\phi \]

\[ = e^{i\phi}\frac{\pi}{2} \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \frac{1}{z^{F_{1, \delta, \rho}}(a, b; a + \alpha + 1)}. \]
where \( \text{Re}(\alpha) > 0 \) and \( \text{Re}(\beta) > 0 \) and \( _2F_{1,p,q} \) is extended Gauss hypergeometric function in Choi et al., [11].

3. Putting \( \delta = \eta = \rho = \tau = 1 \) and \( p = q \) in equation (8), yields
\[
\int_0^\pi e^{i(\alpha+\beta)\phi} (\sin\phi)^{\alpha-1}(\cos\phi)^{\beta-1} _2F_{1,p} \left( a, b; c; e^{i\left(\phi-\frac{\pi}{2}\right)}\sin\phi \right) d\phi
= e^{i\alpha\pi} \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+\beta)} _3F_{2,p} \left( a, b, \alpha; c, \alpha + \beta; 1 \right),
\]
where \( \text{Re}(\alpha) > 0 \) and \( \text{Re}(\beta) > 0 \) and \( _2F_{1,p,q} \) is extended Gauss hypergeometric function in Chaudhry et al., [12].

Setting \( \delta = \eta = \rho = \tau = 1 \) and \( p = q = 0 \) in equation (8), we have
\[
\int_0^\pi e^{i(\alpha+\beta)\phi} (\sin\phi)^{\alpha-1}(\cos\phi)^{\beta-1} _2F_{1} \left( a, b; c; e^{i\left(\phi-\frac{\pi}{2}\right)}\sin\phi \right) d\phi
= e^{i\alpha\pi} \frac{\Gamma(a)\Gamma(c)}{\Gamma(a+\beta)} _3F_{2} \left( a, b, \alpha; c, \alpha + \beta; 1 \right),
\]
where \( \text{Re}(\alpha) > 0 \) and \( \text{Re}(\beta) > 0 \) and \( _2F_{1} \) is extended Gauss hypergeometric function in [13].

6. Conclusions
The following integral that contains generalized hypergeometric function is evaluated
\[
\int_0^\pi e^{i(\alpha+\beta)\phi} (\sin\phi)^{\alpha-1}(\cos\phi)^{\beta-1} _2F_{1,\delta,\eta,\rho} \left( a, b; c; e^{i\left(\phi-\frac{\pi}{2}\right)}\sin\phi \right) d\phi
\]
where \( \text{Re}(\alpha) > 0 \) and \( \text{Re}(\beta) > 0 \). We obtain other interesting results as a special cases of our main result.

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