Annals of Pure and Applied Mathematics
Vol. 25, No. 2, 2022, 91-95
ISSN: 2279-087X (P), 2279-0888(online)
Published on 14 June 2022
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v25n2a05860

Annals of **Pure and Applied Mathematics**

Short communication

An Integral Involving a Generalized Hypergeometric Function

Naresh Dudi [*] and ***Umar Muhammad Abubakar***

[1]Department of Mathematics and Statistics, Jai Narain Vyas University
Jodhpur-342001, India, E-mail: nareshdudi@yahoo.co.in

[2]Department of Mathematics, Faculty of Computing and Mathematical Sciences
Kano University of Science and Technology, Wudil P.M.B, 3244 Kano
Kano State -Nigeria. E-mail: umabubakar347@gmail.com
[*]Corresponding author.

Received 1 May 2022; accepted 8 June 2022

Abstract. In 1961, MacRobert in his very popular, useful and interesting research paper obtained a new type of finite integrals and used the integrals to evaluate integral involving E-functions which he had developed and is a generalization of hypergeometric and generalized hypergeometric functions. The main objective of this short research paper is to find an exciting integral associated with a generalized hypergeometric function by using the integrals obtained by MacRobert. The beauty of our results is that they appear on the product of the ratios of gamma functions. It is clear that the integral associated with gamma functions, the results are very useful from the perspective of the point of view of applications. In terms of parameters, one can easily derive the known integrals due to Rathie and the integral given in Mathai and Saxena's book. It is no exaggeration to mention here that, for other integrals, the transformation and summation formulas involve generalized hypergeometric function.

Keywords: Gamma function, Hypergeometric function, Laguerre polynomial, Mittage-Leffler function, Beta function

AMS Mathematics Subject Classification (2010): 33C75

1. Introduction

Fernandez et al., [1] discussed the following bivariate Mittage-Leffler function that occurs naturally in biological engineering, its fractional calculus properties, and its associations with other special functions such as bivariate Laguerre polynomials:

$$E^{\tau}_{\delta,\eta,\rho}(w,y) = \sum_{r,s=0}^{\infty} \frac{(\tau)_{r+s}}{\Gamma(\delta r+\eta s+\rho)} \frac{w^r y^s}{r!s!}\,, \qquad (1)$$

where $\delta,\eta,\rho,\tau,w,y \in \mathbb{C}$ with $Re(\delta) > 0, Re(\eta) > 0$.

As recently as 2021, Abubakar and Dudi [2] used the two-variable Mittage-Leffler function in equation (1) as a kernel to study some properties and applications to statistical and fractional calculus of the following new generalized beta function:

$$B_{\delta,\eta,\rho}^{\tau;p,q}(x,z) = \int_0^1 t^{x-1}(1-t)^{z-1}\, E_{\delta,\eta,\rho}^{\tau}\left(-\frac{p}{t}, -\frac{q}{1-t}\right) dt, \tag{2}$$

where $Re(\delta) > 0, Re(\eta) > 0, Re(\rho) > 0, Re(\tau) > 0$.

They also [2] studied the following generalized Gauss and confluent hypergeometric functions defined by the beta function in equation (2):

$${}_2F_{1;\delta,\eta,\rho}^{\tau;p,q}(a,b;c;z) = \sum_{r=0}^{\infty}(a)_r \frac{B_{\delta,\eta,\rho}^{\tau;p,q}(b+r,c-b)}{B(b,c-b)}\frac{z^r}{r!}, \tag{3}$$

where $p,q \geq 0, |z| < 1, Re(\delta) > 0, Re(\eta) > 0, Re(\rho) > 0, Re(\tau) > 0, Re(c) > Re(b) > 0, Re(a) > 0$, and

$${}_1F_{1;\delta,\eta,\rho}^{\tau;p,q}(b;c;z) = \sum_{r=0}^{\infty} \frac{B_{\delta,\eta,\rho}^{\tau;p,q}(b+r,c-b)}{B(b,c-b)}\frac{z^r}{r!}, \tag{4}$$

where $p,q \geq 0, |z| < 1, Re(\delta) > 0, Re(\eta) > 0, Re(\rho) > 0, Re(\tau) > 0, Re(c) > Re(b) > 0$.

2. Results required

The following interesting integral formulas by MacRobert's [3] will be present in our current investigation:

$$\int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\phi}(sin\phi)^{\alpha-1}(cos\phi)^{\beta-1}d\phi = e^{i\pi\frac{\alpha}{2}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{5}$$

where $Re(\alpha) > 0$ and $Re(\beta) > 0$,

$$\int_0^{\frac{\pi}{2}} \cos[(\alpha+\beta)\phi]\,(sin\phi)^{\alpha-1}(cos\phi)^{\beta-1}d\phi = \cos\left(\pi\ \frac{\alpha}{2}\right)\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{6}$$

where $Re(\alpha) > 0$ and $Re(\beta) > 0$, and

$$\int_0^{\frac{\pi}{2}} \sin[(\alpha+\beta)\phi]\,(sin\phi)^{\alpha-1}(cos\phi)^{\beta-1}d\phi = \sin\left(\pi\ \frac{\alpha}{2}\right)\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{7}$$

where $Re(\alpha) > -1$ and $Re(\beta) > 0$.

This paper is motivated by the work of several researchers (see for example, [4]-[10]) who have studied many integrals formulas and transforms involving a wide range of special functions of mathematical physics play an important role in many field such as engineering, science and technology.

3. Main result

The following integral involving a generalized hypergeometric function will be evaluated in this article:

$$\int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\phi}(sin\phi)^{\alpha-1}(cos\phi)^{\beta-1}\ {}_2F_{1;\delta,\eta,\rho}^{\tau;p,q}\left(a,b;c;e^{i\left(\phi-\frac{\pi}{2}\right)}sin\phi\right)d\phi$$
$$= e^{i\pi\frac{\alpha}{2}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\ {}_3F_{2;\delta,\eta,\rho}^{\tau;p,q}(a,b,\alpha;c,\alpha+\beta;1), \tag{8}$$

where $Re(\alpha) > 0$ and $Re(\beta) > 0$.

Proof: Expressing the left-hand side of (8) by I, we have

$$I = \int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\phi} (sin\phi)^{\alpha-1} (cos\phi)^{\beta-1} \sum_{r=0}^{\infty} (a)_r \frac{B_{\delta,\eta,\rho}^{\tau;p,q}(b+r,c-b)}{B(b,c-b)r!} e^{in\left(\phi-\frac{\pi}{2}\right)} (sin\phi)^n d\phi.$$

Changing the order of integration and summation (which is reasonably easy seen to justified given the uniform convergence of the series involved in the process), after a little simplification, gives

$$I = \sum_{r=0}^{\infty} (a)_r \frac{B_{\delta,\eta,\rho}^{\tau;p,q}(b+r,c-b)}{B(b,c-b)r!} e^{-in\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta+n)\phi} (sin\phi)^{\alpha+n-1} (cos\phi)^{\beta-1} d\phi \tag{9}$$

Evaluating the integral using MacRobert's (5), we get

$$I = \sum_{r=0}^{\infty} (a)_r \frac{B_{\delta,\eta,\rho}^{\tau;p,q}(b+r,c-b)}{B(b,c-b)r!} e^{-in\frac{\pi}{2}} \left\{ e^{i\pi\frac{\alpha}{2}} \frac{\Gamma(\alpha+n)\,\Gamma(\beta)}{\Gamma(\alpha+n+\beta)} \right\}.$$

And after some simplification, yields

$$I = e^{i\pi\frac{\alpha}{2}} \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{r=0}^{\infty} \frac{(a)_r (\alpha)_n}{(\alpha+\beta)_n} \frac{B_{\delta,\eta,\rho}^{\tau;p,q}(b+r,c-b)}{B(b,c-b)} \frac{1^r}{r!},$$

Finally summing up the series, we easily go to the right-hand side of the (8). Consider equations (6) and (7), which leads to the following result:

Corollary 3.1.

$$\int_0^{\frac{\pi}{2}} \cos[(\alpha+\beta)\phi]\, (sin\phi)^{\alpha-1} (cos\phi)^{\beta-1}\, {}_2F_{1;\delta,\eta,\rho}^{\tau;p,q}\left(a,b;c;e^{i\left(\phi-\frac{\pi}{2}\right)} sin\phi\right) d\phi$$
$$= \cos\left(\pi\,\frac{\alpha}{2}\right) \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\, {}_3F_{2;\delta,\eta,\rho}^{\tau;p,q}(a,b,\alpha;c,\alpha+\beta;1), \tag{10}$$

where $Re(\alpha) > 0$ and $Re(\beta) > 0$, and

$$\int_0^{\frac{\pi}{2}} \sin[(\alpha+\beta)\phi]\, (sin\phi)^{\alpha-1} (cos\phi)^{\beta-1}\, {}_2F_{1;\delta,\eta,\rho}^{\tau;p,q}\left(a,b;c;e^{i\left(\phi-\frac{\pi}{2}\right)} sin\phi\right) d\phi$$
$$= \sin\left(\pi\,\frac{\alpha}{2}\right) \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\, {}_3F_{2;\delta,\eta,\rho}^{\tau;p,q}(a,b,\alpha;c,\alpha+\beta;1), \tag{11}$$

where $Re(\alpha) > -1$ and $Re(\beta) > 0$.

5. Special cases

In this section we will cover some of the very interesting known special cases of our main result.

1. If we take $a = \alpha + \beta$ in equation (8) we get
$$\int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\phi} (sin\phi)^{\alpha-1} (cos\phi)^{\beta-1}\, {}_2F_{1;\delta,\eta,\rho}^{\tau;p,q}\left(a,b;c;e^{i\left(\phi-\frac{\pi}{2}\right)} sin\phi\right) d\phi$$
$$= e^{i\pi\frac{\alpha}{2}} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\, {}_2F_{1;\delta,\eta,\rho}^{\tau;p,q}(a,\alpha;c;1).$$

2. If $\delta = \eta = \rho = \tau = 1$ in equation (8), gives
$$\int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\phi} (sin\phi)^{\alpha-1} (cos\phi)^{\beta-1}\, {}_2F_{1;p,q}\left(a,b;c;e^{i\left(\phi-\frac{\pi}{2}\right)} sin\phi\right) d\phi$$
$$= e^{i\pi\frac{\alpha}{2}} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\, {}_3F_{2;p,q}(a,b,\alpha;c,\alpha+\beta;1),$$

where $Re(\alpha) > 0$ and $Re(\beta) > 0$ and ${}_2F_{1;p,q}$ is extended Gauss hypergeometric function in Choi et al., [11].

3. Putting $\delta = \eta = \rho = \tau = 1$ and $p = q$ in equation (8), yields

$$\int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\phi}(sin\phi)^{\alpha-1}(cos\phi)^{\beta-1}\ {}_2F_{1;p}\left(a,b;c;e^{i\left(\phi-\frac{\pi}{2}\right)}sin\phi\right)d\phi$$

$$= e^{i\pi\frac{\alpha}{2}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\ {}_3F_{2;p}(a,b,\alpha;c,\alpha+\beta;1),$$

where $Re(\alpha) > 0$ and $Re(\beta) > 0$ and ${}_2F_{1;p,q}$ is extended Gauss hypergeometric function in Chaudhry et al., [12].

Setting $\delta = \eta = \rho = \tau = 1$ and $p = q = 0$ in equation (8), we have

$$\int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\phi}(sin\phi)^{\alpha-1}(cos\phi)^{\beta-1}\ {}_2F_1\left(a,b;c;e^{i\left(\phi-\frac{\pi}{2}\right)}sin\phi\right)d\phi$$

$$= e^{i\pi\frac{\alpha}{2}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\ {}_3F_2(a,b,\alpha;c,\alpha+\beta;1),$$

where $Re(\alpha) > 0$ and $Re(\beta) > 0$ and ${}_2F_1$ is extended Gauss hypergeometric function in [13].

6. Conclusions

The following integral that contains generalized hypergeometric function is evaluated

$$\int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\phi}(sin\phi)^{\alpha-1}(cos\phi)^{\beta-1}\ {}_2F_{1;\delta,\eta,\rho}^{\tau;p,q}\left(a,b;c;e^{i\left(\phi-\frac{\pi}{2}\right)}sin\phi\right)d\phi$$

where $Re(\alpha) > 0$ and $Re(\beta) > 0$. We obtain other interesting results as a special cases of our main result.

Acknowledgments. The author is very thankful to the referees for his valuable suggestions which resulted in the improvement of the paper.

Conflict of interest. The authors declare that they have no conflict of interest.

Authors' Contributions. All the authors contributed equally to this work.

REFERENCES

1. A.Fernandez, C.Kurt and M.A.Ozarslan, A naturally emerging bivariate Mittage-Leffler function and associated fractional-calculus operator, *arXiv.org*, 1-22, (2020). https://arXiv.org/abs/2002.12171v1.
2. U.M.Abubakar and N.Dudi, The new generalized beta and associated functions defined by considering bivariate Mittage-Leffler functions regularizer with their applications to distribution theory and fractional calculus, *ISPEC International Congress on Multidisplinary Studies*, November 12-13, 2021/ Adama, Turkey, (2021) 359-374.
3. T.M.MacRobert, Beta functions formula and integral involving E-function. *Math. Annalen*, 143 (1961) 450-452.
4. M.Ali, W.A.Khan and I.Khan, Study on double integral operator associated with generalized Bessel-Maitland function, *Palestine Journal of Mathematics*, 9 (2) (2020) 991-998.

5. S.Mubeen, R.S.Ali, I.Nayab, G.Rahman, T.Abdeljawad and K.S.Nisar, Integral transforms of an extended generalized multi-index Bessel function, *AIMS Mathematics*, 5(6) (2020) 7531-7547.
6. U.M.Abubakar, Some results on generalized Euler-type integral related to the four parameters Mittage-Leffler function, *Journal of New Results in Science*, 10(3) (2021) 1-10.
7. N.U.Khan, T.Usman and M.Gyayasuddin, A note on integral transforms associated with Humbert's confluent hypergeometric function, *Electronic Journal of Mathematical Analysis and Applications*, 4 (2) (2016) 259-265.
8. I.Kim, S.K.Kumari and Y.Vyas, On a new class of integral involving generalized hypergeometric function $_3F_2$, *Homan Mathematical Journal*, 40(10) (2018) 61-73.
9. I.Kim, On a new class of integrals involving generalized hypergeometric function $_4F_3$, *TWMS Journal of Applied and Engineering Mathematics* V, 10(4) (2020) 1049-1054.
10. J.Choi, S.K.Kurumujji, A.Kilicman and A.K.Rathie, Cetain integral formulae associated with the product of generalized hypergeometric series and several elementary functions derived from formulas for the beta function, *Symmetry*, 14 (389) (2022) 1-13.
11. J.Choi, A.K.Rathie and R.K.Parmar, Extension of extended beta, hypergeometric and confluent hypergeometric functions, *Honam Mathematical Journal*, 36(2) (2014) 357-385.
12. M.A.Chaudhry, A.Qadir, H.M.Srivastava and R.B.Paris, Extended hypergeometric and confluent hypergeometric function, *Applied Mathematics and Computation*, 159 (2004) 589-604.
13. E.D.Rainville, *Special functions*, The MacMillan Company, New York, (1960).