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On the Exponential Diophantine Equation 5^x - $2.3^y = z^2$

Sutthiwat Thongnak[1*], ***Wariam Chuayjan***[2] **and** ***Theeradach Kaewong***[3]

[1,2,3]Department of Mathematics and Statistics, Thaksin University
Phatthalung 93210, Thailand.
[2]Email: cwariam@tsu.ac.th; [3]Email: theeradachkaewong@gmail.com
[*]Corresponding author. [1]Email: tsutthiwat@tsu.ac.th

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Abstract. In this article, we study and establish one theorem of the exponential Diophantine equation $5^x - 2\cdot 3^y = z^2$ where x, y and z are non-negative integers. The study reveals that the equation is solvable.

Keywords: exponential Diophantine equation; Number theory

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

A challenging problem for mathematical researchers is the exponential Diophantine equation. They were interested in only integer solution, and tried to find how many solutions there were. In 1844, Eugne Charles Catalan suggested that the only one solution in natural number of $a^x - b^y = 1$ with $a, b, x, y > 1$ is $(a,b,x,y) = (3,2,2,3)$. This conjecture was proved by Mihailescu [5] in 2004. In the last decade, a number of researches involving the exponential Diophantine equation have been studied. For example, Acu [1] proved that $2^x + 5^y = z^2$ has two non-negative integer solutions, which are $(x,y,z) = (3,0,3)$ and $(2,1,3)$. In 2011, Suvanamanee [10] proved that two exponential Diophantine equations, including $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no solution. From 2012 to 2018, many mathematical articles, which involve the exponential Diophantine equation were released [2-4, 6-9]. In 2019, Thongnak et al. [11] studied the exponential Diophantine equation $2^x - 3^y = z^2$. The result revealed that $(x,y,z) = (1,0,1)$ and $(2,1,1)$ are only two non-negative integers to the equation. Two years later, they also proved that the equation $7^x - 5^y = z^2$ has only one trivial solution equation $(x,y,z) = (0,0,0)$ [12]. In 2022, they also studied the equations $7^x - 2^y = z^2$ and $11 \cdot 3^x + 11^y = z^2$ [13,14]. They found that both equations have a unique solution. The previous works indicate that the proof of individual equation is necessary because there is no general method to prove the class of the equation. In this paper, we studied the non-negative solution of the exponential Diophantine equation $5^x - 2\cdot 3^y = z^2$. The basic knowledge in Number theory is applied to obtain the solution.

2. Preliminary

Lemma 2.1. If x is integer, then $x^2 \equiv 0 \bmod 4$ or $x^2 \equiv 1 \bmod 4$.

Proof: Let x be an integer. There is, $q \in \mathbb{Z}$ such that $x = 4q + r$ where $r = 0,1,2,3$. We have $x \equiv r \bmod 4$ which yields $x^2 \equiv r^2 \bmod 4$.

If $r = 0$, we have $x^2 \equiv 0 \bmod 4$.
If $r = 1$, we have $x^2 \equiv 1 \bmod 4$.
If $r = 2$, we have $x^2 \equiv 0 \bmod 4$.
If $r = 3$, we have $x^2 \equiv 1 \bmod 4$.

Therefore, $x^2 \equiv 0 \bmod 4$ or $x^2 \equiv 1 \bmod 4$. □

Lemma 2.2. 3 is not a common divisor of both $5^k - z$ and $5^k + z$ where k and z are non-negative integers.

Proof: We suppose that 3 is the common divisor of both $5^k - z$ and $5^k + z$. It implies that $3 \mid 5^k - z$ and $3 \mid 5^k + z$. Then we have $3 \mid (5^k - z) + (5^k + z)$ or $3 \mid 2\cdot 5^k$. This is a contradiction because $3 \nmid 2\cdot 5^k$. □

Lemma 2.3. If x is an integer, then $x^2 \equiv 0,1 \pmod 3$.

Proof: Let x be any integers. There exists $q \in \mathbb{Z}$ such that $x = 3q + r$ where $r = 0,1$ or 2. This implies that $x \equiv r \pmod 3$ or $x^2 \equiv r^2 \pmod 3$. Since $r^2 = 0,1$ or 4, it has followed that $x^2 \equiv 0,1 \pmod 3$. □

3. Main result

Theorem 3.1. The exponential Diophantine equation $5^x - 2\cdot 3^y = z^2$ where x, y, z are non-negative integers has no solution.

Proof: Let $x, y, z \in \mathbb{Z}^+ \cup \{0\}$ such that

$$5^x - 2\cdot 3^y = z^2. \tag{1}$$

We consider four cases, including the case of $x = y = 0$, case of $x > 0$ and $y = 0$, case of $x = 0$ and $y > 0$, and case of $x, y > 0$.

Case 1: $x = y = 0$ obviously, (1) has no solution because $z^2 = -1$ is impossible.

Case 2: $x > 0$ and $y = 0$ (1) becomes $5^x - 2 = z^2$. Then we have $z^2 \equiv -1 \pmod 4$ or $z^2 \equiv 3 \pmod 4$. By Lemma 2.1, this is impossible.

Case 3: $x = 0$ and $y > 0$ (1) becomes $1 - 2\cdot 3^y = z^2$. This implies that $z^2 < 0$ which is impossible.

Case 4: $x > 0$ and $y > 0$ we obtain that $z^2 \equiv (-1)^x \pmod 3$ from (1), and by Lemma 2.3, this implies that x is even. We let $x = 2k$, for all $k \in \mathbb{Z}^+$. The equation (1) becomes $5^{2k} - z^2 = 2\cdot 3^y$ or

$$(5^k - z)(5^k + z) = 2\cdot 3^y. \tag{2}$$

By Lemma 2.2, we get four possible subcases including, the case of $5^k - z = 1$ and $5^k + z = 2\cdot 3^y$, the case of $5^k - z = 2$ and $5^k + z = 3^y$, the case of $5^k - z = 3^y$ and $5^k + z = 2$, and the case of $5^k - z = 2\cdot 3^y$ and $5^k + z = 1$.

Subcase 4.1: $5^k - z = 1$ and $5^k + z = 2\cdot 3^y$ then we have $2\cdot 3^y + 1 = 2\cdot 5^k$ or $2(5^k - 3^y) = 1$ which is impossible.

Subcase 4.2: $5^k - z = 2$ and $5^k + z = 3^y$ we have $2\cdot 5^k = 2 + 3^y$ or $2(5^k - 1) = 3^y$ which is impossible.

Subcase 4.3: $5^k - z = 3^y$ and $5^k + z = 2$ we have $z = 2 - 5^k$. Since $k > 0$, it yields $z < 0$. This is impossible.

Subcase 4.4: $5^k - z = 2\cdot 3^y$ and $5^k + z = 1$ we have $z = 1 - 5^k$. Obviously, if $k > 0$, then $z < 0$. This is impossible.

Therefore, $5^x - 2\cdot 3^y = z^2$ has no non-negative solution. □

4. Conclusion

We have proved the exponential Diophantine equation $5^x - 2\cdot 3^y = z^2$ where x, y, z are non-negative integers. We have four cases namely: $x = y = 0$, $x > 0$ and $y = 0$, $x = 0$ and $y > 0$, and $x, y > 0$. The basic knowledge used in this proof is given. We have shown that the equation is unsolvable.

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