Annals of Pure and Applied Mathematics
Vol. 26, No. 1, 2022, 1-5
ISSN: 2279-087X (P), 2279-0888(online)
Published on 16 July 2022
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v26n1a01871

Annals of Pure and Applied Mathematics

On the Exponential Diophantine Equation $2^x + 15^y = z^2$

Sutthiwat Thongnak[1*], ***Wariam Chuayjan***[2] **and** ***Theeradach Kaewong***[3]

[1,2,3]Department of Mathematics and Statistics, Thaksin University
Phatthalung 93210, Thailand
[2]Email: cwariam@tsu.ac.th; [3]Email: theeradachkaewong@gmail.com
[*]Corresponding author. [1]Email: tsutthiwat@tsu.ac.th

Received 3 June 2022; accepted 5 July 2022

Abstract. In this article, we solve the exponential Diophantine equation $2^x + 15^y = z^2$ where x, y and z are non-negative integers. The basic theorems in Number theory are given and applied to find all solutions. The result reveals that there are only three solutions to the equation.

Keywords: Diophantine equation; Number theory

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

A popular topic in Number theory is the Diophantine equation. It is known as one equation, which has two or more unknown variables, and the only solutions of interest are integers. Over a decade, many mathematicians have studied the exponential Diophantine equation in the form $a^x + b^y = z^2$ where a and b are positive integers and x, y, z are non-negative integers. The example articles involving the equation are as follows. In 2007, Acu [1] proved that the equation with $a = 2$ and $b = 5$ has only two solutions which are $(x, y, z) \in \{(3,0,3),(2,1,3)\}$. Four years later, Suvarnamani et al. [13] presented two equations with $a = 4$ and $b = 7$ or 11. They proved that both equations have no non-negative solution. From 2012 to 2020, many equations in the form $a^x + b^y = z^2$ were studied. Some equations were solvable [2, 3, 7, 9-11], but some equations were not [4, 5, 12-14]. In 2021, there are many mathematicians the Diophantine equation in this form. For example, Dokchan and Pakapongpun presented [6] the Diophantine equation $p^x + (p+20)^y = z^2$ where p and $p+20$ are primes. They proved that $3^x + 23^y = z^2$, has no solution, and applied this result to prove that $p^x + (p+20)^y = z^2$ has no solution. In the same year, Paisal and Chayapham [9] studied $17^x + 83^y = z^2$ and $29^x + 71^y = z^2$. They proved that two equations have the same solution, which is $(x, y, z) = (1,1,10)$. In 2022, Thonagnak el al. [15] proved that $11 \cdot 3^x + 11^y = z^2$ has the unique solution that is

$(x,y,z)=(2,0,10)$. At present, there is no general method to prove the class of this equation. The proof of individual equations still need. In this article, we study the equation $a^x+b^y=z^2$ for $a=2$ and $b=15$. The basic theorems in Number theory and Catalan's conjecture are applied to determine and prove all non-negative integer solutions.

2. Preliminary

Lemma 2.1. (Catalan's conjecture) [8] Let a,b,x and y are integers. The Diophantine equation $a^x-b^y=1$ with $\min\{a,b,x,y\}>1$ has a unique solution $(a,b,x,y)=(3,2,2,3)$.

Lemma 2.2 5 is not a common divisor of both $(z+2^k)$ and $(z-2^k)$ where z,k be non-negative integers.

Proof: Let 5 be the common divisor of both $(z+2^k)$ and $(z-2^k)$. This implies that $5|(z-2^k)$ and $5|(z+2^k)$. We can write $5|(z+2^k)-(z-2^k)$ or $5|2^{k+1}$. This is impossible. □

Lemma 2.3 If x is an integer, then $x^2\equiv 0,1 \bmod 3$.

Proof: Let x be any integers. There exists $q\in\mathbb{Z}$ such that $x=3q+r$ where $r=0,1$ or 2. This implies that $x\equiv r \pmod 3$ or $x^2\equiv r^2 \pmod 3$. Since $r^2=0,1$ or 4, it follows that $x^2\equiv 0,1 \pmod 3$. □

3. Main result

Theorem 3.1. Let $x,y,z\in\mathbb{Z}^+\cup\{0\}$. The exponential Diophantine equation $2^x+15^y=z^2$ have only three solutions, which are $(x,y,z)=(3,0,3),(0,1,4)$ and $(6,2,17)$.

Proof: Let x,y and z be non-negative integers such that

$$2^x+15^y=z^2 . \tag{1}$$

We consider four cases, including the case of $x=y=0$, the case of $x>0$ and $y=0$, the case of $x=0$ and $y>0$, and the case of $x>0$ and $y>0$.

Case 1: $x=y=0$ we have $z^2=2$ by (1). This is impossible.

Case 2: $x>0$ and $y=0$ from (1), we have

$$z^2-2^x=1. \tag{2}$$

If $x=1$, then (2) becomes $z^2=3$. This is impossible. If $x\geq 2$, then it implies that $z\geq 2$. By Lemma 2.1, we obtain $x=3$ and $z=3$. Hence $(x,y,z)=(3,0,3)$ is the solution of (1).

Case 3: $x=0$ and $y>0$ from (1), we have

$$z^2 - 15^y = 1 \ . \qquad (3)$$

If $y = 1$, then (3) becomes $z^2 = 16$. Thus, we must have $z = 4$. From this case, we obtain that $(x, y, z) = (0, 1, 4)$ is the solution of (1). If $y > 1$, then (3) has no solution by Lemma 2.1.

Case 4: $x > 0$ and $y > 0$ we consider two subcases, which are $x = 1$ and $x > 1$.

Subcase 4.1: $x = 1$ from (1), we have $2 + 15^y = z^2$. Because of $15^y \equiv 0 \pmod 3$, this implies that $z^2 \equiv 2 \pmod 3$. By Lemma 2.3, this is impossible.

Subcase 4.2: $x > 1$. From (1), we have $(-1)^x \equiv z^2 \pmod 3$. Then x is even. Let $x = 2k$, for all $k \in \mathbb{Z}^+$. From (1), we have $15^y = z^2 - 2^{2k}$ or

$$3^y \cdot 5^y = (z - 2^k)(z + 2^k). \qquad (4)$$

This implies that $3 \mid (z - 2^k)(z + 2^k)$. It can be seen that $3 \mid z - 2^k$ or $3 \mid z + 2^k$. We consider for $3 \mid z - 2^k$ and $3 \mid z + 2^k$.

The case of $3 \mid z - 2^k$, it follows that $3 \nmid z + 2^k$. By (4) and Lemma 2.2, we have two possible cases, namely: $z - 2^k = 3^y$, $z + 2^k = 5^y$ and $z - 2^k = 3^y \cdot 5^y$, $z + 2^k = 1$.

i) For $z - 2^k = 3^y$, $z + 2^k = 5^y$, then we have

$$5^y - 3^y = 2^{k+1} \ . \qquad (5)$$

(5) implies that $1 - (-1)^y \equiv 0 \pmod 4$, so y must be even. Then we let $y = 2t$, for all $t \in \mathbb{Z}^+$, thus (5) becomes $5^{2t} - 3^{2t} = 2^{k+1}$ or $(5^t - 3^t)(5^t + 3^t) = 2^{k+1}$. There are $\alpha, \beta \in \mathbb{Z}$ which are $5^t - 3^t = 2^\alpha$ and $5^t + 3^t = 2^\beta$ where $0 \le \alpha < \beta$ and $\alpha + \beta = k + 1$. Then we have $2 \cdot 5^t = 2^\alpha + 2^\beta$ or $2 \cdot 5^t = 2^\alpha (1 + 2^{\beta - \alpha})$. It can be seen that $\alpha = 1$ and $5^t = 1 + 2^{\beta - 1}$. We have

$$5^t - 2^{\beta - 1} = 1. \qquad (6)$$

If $\beta = 1$, then (6) becomes $5^t = 2$. This is impossible. If $\beta = 2$, then (6) becomes $5^t = 3$. This is impossible. If $\beta = 3$, then (6) becomes $5^t = 5$. We obtain $t = 1$, and it follows that $\alpha = 1$ and $k = 3$. Consequently, we get $(x, y, z) = (6, 2, 17)$ is a solution to the equation (1). If $\beta \ge 4$, by Lemma 2.1 (6) has no solution .

ii) For $z - 2^k = 3^y \cdot 5^y$, $z + 2^k = 1$ because $z - 2^k < z + 2^k$, $3^y \cdot 5^y < 1$. This is impossible.

The case of $3 \mid z + 2^k$ obviously, $3 \nmid z - 2^k$. By (4) and Lemma 2.2, we have two possible cases namely: $z + 2^k = 3^y$, $z - 2^k = 5^y$ and $z + 2^k = 3^y \cdot 5^y$, $z - 2^k = 1$.

i) For $z + 2^k = 3^y$, $z - 2^k = 5^y$, this implies that $z + 2^k < z - 2^k$ or $2 \cdot 2^k < 0$ which is impossible.

ii) For $z+2^k=3^y\cdot 5^y$, $z-2^k=1$, we have $15^y-1=2^{k+1}$. This implies that $2^{k+1}\equiv 0\pmod 7$, so $7\,|\,2^{k+1}$ which is impossible.

Hence, the exponential Diophantine equation has three solutions which are $(x,y,z)\in\{(3,0,3),(0,1,4),(6,2,17)\}$. □

4. Conclusion

In this study, we have applied the basic theorems in Number theory such as the factoring method, modular method, and Catalan's conjecture to solve the exponential Diophantine equation $2^x+15^y=z^2$. We have found three non-negative integer solutions $(x,y,z)\in\{(3,0,3),(0,1,4),(6,2,7)\}$.

Acknowledgments. We thank reviewers for their careful reading of our manuscript and the useful comments.

Conflict of interest. The authors declare that they have no conflict of interest.

Authors' Contributions. All the authors contributed equally to this work.

REFERENCES

1. D.Acu, On a Diophantine equation $2^x+5^y=z^2$. *General Mathematics*, 15(4) (2007) 145-148.
2. S.Asthana, On the Diophantine equation $8^x+113^y=z^2$, *International Journal of Algebra*, 11 (2017) 225-230.
3. S.Asthana and M.M.Singh, On the Diophantine equation $3^x+117^y=z^2$, *GANITA*, 70(2) (2020) 43-47.
4. N.Burshtein, On Solution to the Diophantine equations $5^x+103^y=z^2$ and $5^x+11^y=z^2$ with Positive Integers x,y,z, *Annals of Pure and Applied Mathematics*, 19(1) (2019) 75-77.
5. N.Burshtein, On two Diophantine equations $2^x+5^y=z^2$ and $7^x+11^y=z^2$, *Annals of Pure and Applied Mathematics*, 21(1) (2020) 63-68.
6. R.Dokchan and A.Pakapongpun, On the Diophantine equation $p^x+(p+20)^y=z^2$ where p and $p+20$ are primes, *International Journal of Mathematics and Computer Sciences*, 16(1) (2021) 179-183.
7. J.Fergy and T.Rabago, On the Diophantine equation $2^x+17^y=z^2$, *Journal of the Indonesian Mathematical Society*, 22(2) (2016) 85-88.
8. P.Mihailescu, Primary cyclotomic units and a proof of Catalan's conjecture, J. Reine Angew Math., 572 (2004) 167 – 195.
9. K.Paisal and P.Chayapham, On exponential Diophantine equation $17^x+83^y=z^2$ and $29^x+71^y=z^2$, *Journal of Physics: Conference Series*, (2021).

10. B.Sroysang, On the Diophantine equation $3^x + 17^y = z^2$, *International Journal of Pure and Applied Mathematics*, 89(1) (2013) 111-114.
11. B.Sroysang, More on the Diophantine equation $2^x + 3^y = z^2$, *International Journal of Pure and Applied Mathematics*, 84(2) (2013) 133-137.
12. B.Sroysang, On two Diophantine equations $7^x + 19^y = z^2$ and $7^x + 91^y = z^2$, *International Journal of Pure and Applied Mathematics*, 92(1) (2014) 113-116.
13. A.Suvarnamani, A.Singta and S.Chotchaisthit, On two diophantine equation $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$. *Science and Technology RMUTT Journal*, 1(1) (2011) 25-28.
14. S.Thongnak, T.Kaewong and W.Chuayjan, On the Exponential Diophantine Equation $(p+2)^x + (5p+6)^y = z^2$ when $p, p+2$ and $5p+6$ are Primes, *Annals of Pure and Applied Mathematics*, 23(2) (2021) 111-115.
15. S.Thongnak, W.Chuayjan and T.Kaewong, On the Diophantine equation $11 \cdot 3^x + 11^y = z^2$ where x, y and z are non-negative integers, *Annals of Pure and Applied Mathematics*, 25(1) (2022) 51-54.