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Brief Note

Sufficient Conditions for Hamiltonian Fuzzy Graphs

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Abstract. In this note, we present sufficient conditions for Hamiltonian cycles in fuzzy graphs. In particular, we extend the well-known Ore's theorem and Dirac's theorem in graph theory to fuzzy graph theory.

Keywords: Hamiltonian cycle, fuzzy graph

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1. Introduction

We call the graphs in the graph theory as crisp graphs. We use the notation and terminology [1] when we deal with crisp graphs. We use n to denote the number of vertices in a crisp graph. A cycle C in a crisp graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A crisp graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a crisp graph G is called a Hamiltonian path of G if P contains all the vertices of G. A standard definition of a fuzzy graph is as follows (see, for example, [2]). Let G = (V, E) be a crisp graph with $V \neq \emptyset$.

Definition 1.1. A fuzzy graph G[f] = (V, σ, μ) based upon the crisp graph G is G together with a pair of functions $\sigma : V \rightarrow [0, 1]$ and $\mu : V \times V \rightarrow [0, 1]$ such that for all a, b in V, $\mu(a, b) \leq \min\{\sigma(a), \sigma(b)\}$, where σ (a), σ(b), and μ(a, b) represent the membership values of the vertex a, vertex b, and edge (a, b) in G[f], respectively.

Notice that, in general, (a, b) and (b, a) (which are in V x V) are not the same in Definition 1.1, where a and b are two distinct elements in V. Following Definition 1, we can introduce the following concept for the fuzzy graphs in which there is no distinction between (a, b) and (b, a) in new fuzzy graphs.

Definition 1.2. Let V be a vertex set of a crisp graph G with $|V| \geq 1$ and $V^{[2]}$ be the set of all the 2-element multisubsets of V. Namely, $V^{[2]} = \{\{a, b\} : a \in V, b \in V, a \neq b\} \cup \{\{a, a\} : a \in V\}$. A fuzzy graph G[f] = (V, σ, μ) based upon the crisp graph G is G together with a pair of functions $\sigma : V \rightarrow [0, 1]$ and $\mu : V^{[2]} \rightarrow [0, 1]$ such that $\mu(\{a, b\}) \leq \min\{\sigma(a), \sigma(b)\}$ for each $\{a, b\} \in V^{[2]}$ with $a \neq b$ and $\mu(\{a, b\}) = 0$ with a = b, where σ(a),

$\sigma(b)$, and $\mu(a, b)$ represent the membership values of the vertex a, vertex b, and edge (a, b) in G[f], respectively.

Notice that, in Definition 1.2, we can believe that vertex a and vertex b are not adjacent in G[f] if $\mu(\{a, b\}) = 0$. This implies that G[f] does not have any self-loops since $\mu(\{a, a\}) = 0$ for each vertex in G[f]. If G = (V, E) is a crisp simple undirected graph with $|V| \geq 1$, then G can be thought as a fuzzy graph if we let $\sigma(a) = 1$ for each vertex a in V and $\mu(e) = 1$ for each edge e in E in Definition 1.2. So Definition 1.2 is a generalization of crisp simple undirected graphs with $|V| \geq 1$. All the fuzzy graphs to be discussed in this note are the ones defined in Definition 1.2.

The degree of a vertex u in G[f], denoted $d_{G[f]}(u)$, is defined as $\sum_{b \neq u} \mu(u, b)$ (see Definition 1.13 on Page 10 in [2]). A Hamiltonian cycle in $G[f] = (V, \sigma, \mu)$ with $n := |V| \geq 3$ is a sequence of vertices $a_1, a_2, ..., a_n, a_1$ such that $a_i \neq a_j$ with $1 \leq i \neq j \leq n$ and $\mu(\{a_i, a_{i+1}\}) > 0$ with $1 \leq i \leq n$, where the index (n + 1) is regarded as 1. A fuzzy graph $G[f] = (V, \sigma, \mu)$ is Hamiltonian if $G[f] = (V, \sigma, \mu)$ has a Hamiltonian cycle. Notice that if $\sigma(a) = 0$ then $\mu(\{a, b\}) = 0$ for each b in V, where $b \neq a$. Then if a fuzzy graph $G[f] = (V, \sigma, \mu)$ is Hamiltonian then $\sigma(a) > 0$ for each a in V. A Hamiltonian path in $G[f] = (V, \sigma, \mu)$ with $n := |V| \geq 1$ is a sequence of vertices $a_1, a_2, ..., a_n$ such that $a_i \neq a_j$ with $1 \leq i \neq j \leq n$ and $\mu(\{a_i, a_{i+1}\}) > 0$ for each i with $1 \leq i \leq (n - 1)$.

For a fuzzy graph $G[f] = (V, \sigma, \mu)$ with $\mu(\{u, v\}) = 0$ and $u \neq v$, the fuzzy graph $(G + uv)[f] = (V_1, \sigma_1, \mu_1)$ is defined as follows: $V_1 = V$, $\sigma_1 = \sigma$, $\mu_1(\{x, y\}) = \mu(\{x, y\})$ if $\{x, y\}$ is in $V^{[2]} - \{u, v\}$ and $\mu_1(\{u, v\})$ is any real number r such that $0 < r \leq 1$ and $r \leq \min\{\sigma(u), \sigma(v)\}$.

In this note, we present sufficient conditions for Hamiltonian fuzzy graphs. The main result is as follows.

Theorem 1.1. Let $G[f] = (V, \sigma, \mu)$ be a fuzzy graph with $n := |V| \geq 3$ and $\sigma(u) > 0$ for each vertex u in V. If $d_{G[f]}(u) + d_{G[f]}(v) \geq n$ for an element {u, v} in $V^{[2]}$ with $u \neq v$ and $\sigma(\{u, v\}) = 0$, then $G[f] = (V, \sigma, \mu)$ is Hamiltonian if and only if $(G + uv)[f] = (V_1, \sigma_1, \mu_1)$ is Hamiltonian.

2. Proofs of Theorems 1.1

Proof of Theorem 1.1. It is obvious that (G + uv)[f] is Hamiltonian if G[f] is Hamiltonian. Now suppose $(G + uv)[f] = (V_1, \sigma_1, \mu_1)$ is Hamiltonian. If $G[f] = (V, \sigma, \mu)$ is not Hamiltonian, then $G[f] = (V, \sigma, \mu)$ has a Hamiltonian path $u := a_1, a_2, ..., a_n := v$ such that $\mu(\{a_i, a_{i+1}\}) > 0$ for each i with $1 \leq i \leq (n - 1)$. Set

$$S := \{a_i : \mu(\{u, a_i\}) > 0\}, \ \ T := \{a_{i+1} : \mu(\{v, a_i\}) > 0\}.$$

Obviously, u is not in T. Since $\mu(\{u, u\}) = 0$, u is not in S. Thus $|S \cup T| < n$. We further have that $S \cap T$ is empty. Suppose, to the contrary, that $a_j \in S \cap T$. Then $a_j \neq v$ since $\mu(\{u, v\}) = 0$. Thus G[f] has a Hamiltonian cycle

$$a_1, a_2, ..., a_{j-1}, a_n, a_{n-1}, ... , a_j, a_1,$$

a contradiction. Now we have the following final contradiction:

$n \leq d_{G[f]}(u) + d_{G[f]}(v) = \sum_{b \neq u} \mu(u, b) + \sum_{b \neq v} \mu(v, b)$

$\leq |S| + |T| = |S \cup T| + |S \cap T| < n.$

The proof of Theorem 1.1 is complete. □

Clearly, Theorem 1.1 has the following Corollary 2.1 and Corollary 2.1 implies Corollary 2.2 below.

Corollary 2.1. Let $G[f] = (V, \sigma, \mu)$ be a fuzzy graph with $n := |V| \geq 3$ and $\sigma(u) > 0$ for each vertex u in V. If $d_{G[f]}(u) + d_{G[f]}(v) \geq n$ for each $\{u, v\}$ in $V^{[2]}$ with $u \neq v$ and $\mu(\{u, v\}) = 0$ in G[f], then G[f] is Hamiltonian.

Corollary 2.2. Let $G[f] = (V, \sigma, \mu)$ be a fuzzy graph with $n := |V| \geq 3$ and $\sigma(u) > 0$ for each vertex u in V. If $2\min\{d_{G[f]}(u) : u \in V\} \geq n$, then G[f] is Hamiltonian.

It is obvious that Corollary 2.1 and Corollary 2.2 are the generalizations of Ore's Theorem and Dirac's Theorem in the Hamiltonian graph theory, respectively.

3. Conclusion

In this note, we introduced the concept of fuzzy graphs, which generalizes the concept of simple undirected graphs. We also presented a theorem on the Hamiltonian fuzzy graphs. This theorem enables the generalizations of well-known Ore's theorem and Dirac's theorem for crisp graphs to fuzzy graphs.

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