Annals of Pure and Applied Mathematics
Vol. 26, No. 1, 2022, 19-26
ISSN: 2279-087X (P), 2279-0888(online)
Published on 31 August 2022
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v26n1a04879

Annals of
Pure and Applied Mathematics

Review Article

On the Solubility of Polynomials of Degree *n*

Sunday Adesina Adebisi

Department of Mathematics, Faculty of Science, University of Lagos, Nigeria.
Email: adesinasunday@yahoo.com

Received 3 July 2022; accepted 25 August 2022

Abstract. The concepts of solvability in polynomials is highly imperative, most especially the concerned operations on the sums and the products of the roots. In this paper, an analysis of the fundamentals and the basic characterizations of the roots of polynomials was considered. This involves but is not limited to the sums as well as the products of their roots, not only those of the elementary or lower degrees but also that of any higher degree n. Efforts have been intensified to state and prove certain characterizations which each case of the degrees of the polynomial must satisfy. Hence, the analysis further helps in determining of zeros for any given polynomial, including those of higher degrees.

Keywords: Polynomial function, Extention field, Indeterminate, Coefficient matrix.

AMS Mathematics Subject Classification (2010): 11C08, 08A40

1. Introduction

Polynomials of degree n are very valuable in many mathematical operations, computations and general sciences. Results involving polynomials have been found very useful and applicable to physics, chemistry, biological sciences, and even in the real-life situations and circumstances. Many life situations, cases, as well as problems, can easily be modeled mathematically using ideas from polynomials in order to proffer suitable solutions as required.

Given that W is a field for consideration. Choose the variables $x_1, x_2, \cdots, x_n$ over the field W. One of the main and essential objects and entities for studying algebraic geometry as well as commutative algebra is contained in systems of polynomial equations. Such form of polynomial expressions can thus be given as follows :

$$F_1(x_1, x_2, \cdots, x_n) = 0, \cdots, F_m(x_1, x_2, \cdots, x_n) = 0 \qquad (1)$$

In order to find the explicit (set of) solutions to (1), many techniques and methods have been developed (please, see [3]). As a matter of fact, the multivariate resultants methods were introduced by Macaulay. (Please see [4]). These were used in solving the systems of the polynomial equations. This was done by eliminating the variables in turn. Now, given some fields W, to solve the systems of polynomial equations over the finite field W, the Grobner basis methods were used even though the whole structure seems to be somehow complex in forms and seems to be some difficulties trying to comprehend and get proper understanding (see [3]).

The concept of solubility or solvability in polynomials is highly imperative, most especially the concept of sums and products of the roots. In [2]. The whole idea was limited to monic polynomials of degree n, whence a_n = 1. Meanwhile, in this work the general characterizations for all polynomials were examined. This involves but not limited to the sums as well as the products of their roots not only those of the elementary or lower degrees but also that of any higher degree n. Efforts have been intensified to state and prove certain characterizations which each case of the degrees of the polynomial must satisfy. Hence, the analysis further helps in the determination of the zeros for any given polynomial, including those of higher degrees.

2. Basic terminologies and useful ideas from Galois theory

We have the following notable items on algebraic closure, fields and splitting fields.

Definition 1. (see [1]) Suppose that $L: W$ be a field extension, and given that $f \in W[x]$ is a polynomial. If these coefficients are in W, we say that the polynomial f split over the field L. This happens only whenever f is a constant polynomial or if there is the existence of some $\gamma_1, \gamma_2, \cdots \gamma_n$ of $L \ni$
$f(x) = c(x - \gamma_1)(x - \gamma_2) \cdots (x - \gamma_n)$. Here, the leading coefficient of f is $c \in W$.

Definition 2. (see [1]) Given that W is a field. Define an extension $L: W$ of W as an embedding of W in some larger field L. Suppose that $L: W$ is a field extension and let A be any subset of L, we have that the set $W \cup A$ will generate subfield $W(A)$ of L. This is the intersection for al the l subfields of L. These contain $\cup A$ (Here, i should be noted that any intersection for subfields of L will itself also be a subfield of L). and it could be said that $W(A)$ is the field which is obtainable from W and could be by the adjoining of the set A. Also, denote $W(\gamma_1, \gamma_2, \cdots \gamma_k)$ for any finite subset $\{\gamma_1, \gamma_2, \cdots \gamma_n\}$ of L. In particular, $K(\gamma)$ denotes the field obtained by adjoining some element γ of L to W. A field extension $L: W$ is said to be simple if there exists some element γ of L such that $L = W(\gamma)$.

Definition 3. (see [1]) If $L: W$ is a field extension, and suppose that γ is an element of L and if $\exists$ some (non zero) polynomial $f \in W[x]$ which have coefficients $\ni W \ni$ $f(\gamma) = 0$, then we say that γ is algebraic over W. If this is not so, then, it could be said that γ is transcendental over W. Also, we say that a field extension $L: W$ is algebraic if it is certain that every element of L is also algebraic over W as required.

Definition 4 (see [1]) Suppose that $L: W$ is a field extension. Then, L can be said to be such as a vector space over the field W. Now, let L be a finite-dimensional vector space over the field W. Then the extension $L; W$ could be said as finite. Now, define the degree $L: W$ to be the dimension of L, which is considered as a vector space over the field W. For more other details as regards, polynomials and equations may likely be of some importance (please, see [5 to 30]).

Lemma 1. (see [1]) Every finite field extension $[L: W]$ is algebraic.
Proof: Suppose that $L: W$ is a finite field extension. Also, let $n = [L: W]$. be the degree or the order of the field extension. Given that $\gamma \in L.$. We have that, either of the set of

elements $1, \gamma, \gamma^2, \cdots, \gamma^n$ are not all districts or else the elements are of linear dependence over the given field W (By inference, a linearly independent subset of L can be said to possibly have at most n elements).

Hence, $\exists\ r_0, r_1, r_2, \cdots, r_n \in W$, not all zero, in which

$$r_0 + r_1\gamma + r_2\gamma^2 + \cdots + r_n\gamma^n = 0 \qquad (2)$$

Thus, γ is algebraic over W. This shows that the field extension L: W is algebraic as required.□

3. Soluble groups

The notable Mathematician by name Evariste Galois was the individual who introduced the concept of a soluble group into the field of Mathematics. This was actually done so as to state as well as to give me proof of his fundamental general theorems in line with the solvability concerning the polynomial equations.

Definition 5. (see [1] and [2]) A group P can be said to be soluble (also called solvable) if $\exists$ a finite sequence $P_0, P_1, \cdots, P_n$ all of whose are subgroups of P, such that $P_0 = \{1\}$ and $P_n = P$, and $\ni$ P_{i-1} is observed to be normal in P_i and also that P_i/P_{i-1} is also abelian for $i = 1,2, \cdots, n$.

Definition 6. (please see [1]) A commutative ring F with an identity such that every non-zero element of F which is non zero is also a unit is known to be a field. Here, the so called field F, is an additive abelian group.

Definition 7. Every polynomial $P(x)$ in variable x having the coefficients in some field F has an expression which is of the form stated as follows :

$$P(x) = \sum_{i=0}^{n} A_i x^i = A_0 + A_1 x + A_2 x^2 + \cdots + A_{n-1}x^{n-1} + A_n x^n, \qquad (3)$$

Here, $A_i \in F, 0 \le i \le n$

$P(x)$ can be referred to as a polynomial function, and this happens whenever x appears as a variable or indeterminate. A polynomial equation $P(x) = C$ is expressible in a matrix form. This could be given as $P(X) = AX = K$, where the entity A is the coefficient matrix, X, the matrix of the powers of the indeterminate (variable) x and B, a constant matrix.

We thus, have:

$P(x) = [a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_n][1x^2 : x^n] = b \in F$. Here, we multiply a matrix of order $1 \times n$ with another vector matrix of order $n \times 1$, which produces a scalar $b \in F$

Definition 8. Any polynomial of two indeterminates is given by :

$$P(x) = \sum_{i \le j=0}^{n} a_{ij} x_1^i x_2^j. \qquad (4)$$

A polynomial $P(x) = b$ of two indeterminates x, y written as:

$F[x, y]$ (where $b, a_{ij} \in F$) is better expressed in matrix form as:

$$P(x)=\begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0n}a_{10} & a_{11} & a_{12} \vdots & a_{22}a_{n0} & a_{n1} & a_{n2} \\ \cdot & a_{nn} \end{bmatrix}$$
$$\times \begin{bmatrix} 1 & x & x^2 & & \cdots & x^n y & xy & x^2y & \cdots & x^n y y^2 & xy^2 \\ x^2y^2 & \cdots & x^n y^2 \vdots y^n & xy^n & x^2y^n & \cdots & x^n y^n \end{bmatrix}$$
$$=[b_1 b_2 b_3 \vdots b_n]$$
$$= b_1+b_2+\cdots+b_n=b\in F.$$

Theorem 2. (This is a Fundamental Theorem in Algebra) (please see [2])

a) For e very polynomial of a given degree $n\geq 1$ there is at least one zero (root). This is contained in the complex numbers.

b) Let $P(x)$ be the polynomial with degree $n,$ then $P(x)$ can be said to have exactly n district roots. These roots may either be real numbers or complex numbers. (In 1799, Gauss worked on this theorem's proofs for doctoral dissertation topic in his Ph.D.)

Theorem 3. Quoted from [2] (This is the Maximum Number of Zeros (roots) Theorem) (Please see [2]). It's not possible for a polynomial to have more real zeros (roots) more than the degree of the polynomial .

Proof: By contradiction, let P(x) be of $deg;;m\geq 1$. Now, if $t_1,\cdots,t_m,t_{m+1}$ are $m+1$ roots possessed by the polynomial . By using the factor theorem, there exists polynomial $A_1(x)$ which has the $deg <$ that of $deg(P(x))$ by one $\ni$ $P(x)=(x-t_1).A_1(x)$. Now, since $P(t_2)=0$, and t_1,t_2 are distinct, that is , $t_1\neq t_2$, it must be that $A_1(t_2)$ = o. Again, by factor theorem. Write this as , $P(x)=(x-t_1).(x-t_2).A_2(x),$ whence $A_2(x)$ is of $deg2$ less than $P(x)$. Now, since t_3 is also distinct from t_1 and t_2 , we must have $A_2(T_3)$ = 0 . We have $P(x)=(x-t_1).(x-t_2).(x-t_3).A_3(x)$, where $deg(A_3(x))$ is of $deg3$ less than that of $P(x)$. If we continue the process this way and the process is repeated until reaching the given stage for which $P(x)=(x-t_1)\cdots(x-t_m).A_m(x)$, and $A_m(x)$ is of $degm$ less than $P(x)$. Since $P(x)$ only had $degm$ in first , $A_m(x)$ has to be of $deg0$. This makes $A_m(x)$ to be of a constant. Allow the constant to be $A_m(x)=b$. Hence,, t_{m+1} is still a root for $P(x)$, and since t_{m+1} is distinct from all the other t_i, it must be that $A_m(t_{m+1})$ = 0 . This can be possible only when $b=0\Rightarrow P(x)\equiv 0 \Rightarrow\Leftarrow$ since our assumption is that $m\geq 0$. Hence, our initial assumption must be wrong. Thus, we must have that $P(x)$ has roots $\{t_i\}, i\in\{1,2,\cdots,m\}$.□

Theorem 4. (The product and sum theorem) (see [2])

Suppose that $P(x)=x^n+A_{n-1}x^{n-1}+\cdots+A_3x^3A_2x^2+A_1x+A_0=\sum_{k=0}^{n} A_kx^k=$ 0) be a monic polynomial equation with real coefficients, where $n\geq 1$, then a_0 can be calculated as $(-1)^n\times$ the product of the roots found for $P(x)$. Also, a_{n-1} happens to be the opposite of the sum of the roots of $P(x)$ Mathematically, it means that :

(i) $a_0=\Pi_{i=1}^{n} r_i$ and (ii) $a_{n-1}=-\sum_{i=1}^{n} r_i$

Proof: By Theorems 1 and 2, $P(x)$ has n roots. Let These roots be denoted by $r_1,r_2,r_3,\cdots,r_n$. Now, the product for n factors that are associated to the roots can be formed. Now, given that $q[x]=[(x-r_1)(x-r_2)(x-r_3)\cdots(x-r_n)]$. This proof is concluded by multiplying out all these terms and by inspecting the coefficient on x^{n-1} alongside the constant term. On the other hands, by using induction on n,the formal proof can be made thus: Let $n = 1$, we have $P(x) = x + a_0$ and for this situation, the only zero for $P(x)$ will be $r_1 = -a_0$. Now r_1 is only the root, r_1 is itself the product for the roots. Meanwhile, $(-1)^n a_0 = (-1)^1 a_0 = -a_0 = r_1$. Thus, this settles the part involving the constant term. Also, since r_1 is only the root, r_1 is itself the sum of the roots. The second leading coefficient is the opposite of the sum of the roots. This is because $a_0 = -(r_1)$. More instructively, by manually looking at the case when n = 2 before the setting up of the induction step. Also $[(x - r_1)(x - r_2)] = [x^2 + (-r_1 + -r_2)x + r_1 r_2]$. Here as well, it is very clear that the second coefficient which is leading is the opposite of the sum of the roots, the constant is product of the roots. $P(x)$ is ih quadratic form and we have that $n = 2 \Rightarrow (-1)^n - (-1)^2 = 1$. Now let it be that the result is true at anytime that we have k zeros. If $P(x)$ is the polynomial having $k + 1$ zeros, i.e. $P(x) = [(x - r_1)(x - r_2) \cdots (x - r_{k+1})]$. Now consider that we may write: $P(x) = [\{(x - r_1)(x - r_2) \cdots (x - r_k)\}(x - r_{k+1})]$.

Given that $q(x) = \{(x - r_1)(x - r_2) \cdots (x - r_k)\}$. Then $q(x)$ has degree k. The induction hypothesis is thus applicable to $q(x)$. Now, write $q(x) = [x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0]$ then, we know that $a_{k-1} = -(\sum_{i=1}^{k} r_i)$ and we know that $a_0 = (-1)^k . \Pi_{i=1}^{k} r_i$.

Now $P(x) = q(x).(x - r_{k+1}) = (x - r_{k+1}).\{x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0\}$
$= \{x^{k+1} + a_{k-1}x^k + \cdots + a_1 x^2 + a_0 x\} + \{(-r_{k+1})x^k + (-r_{k+1})a_{k-1}x^{k-1} + \cdots + (-r_{k+1})a_1 x + (-r_{k+1})a_0\} = x^{k+1} + \{a_{k-1} + (-r_{k+1})\}x^k + \cdots$
$+(a_0 - r_{k+1}.a_1)x + (-r_{k+1})a_0$.

We have $a_{k-1} + (-r_{k+1}) = -(\sum_{i=1}^{k} r_i) + (-r_{k+1}) = -(\sum_{i=1}^{k+1} r_i)$
and $(-r_{k+1}).a_0 = (-r_{k+1}).(-1)^k.(\Pi_{i=1}^{k} r_i) = (-1)^{k+1}.(\Pi_{i=1}^{k+1} r_i)$ □

4. Materials and methods

Here, the normal, basic mathematical fundamentals are applied in order to prove the results.

Statement of Problem

Proposition 5.

$$Suppose\ \ that \qquad F(x) = \sum_{i=0}^{N} a_i x^i$$

is a polynomial. Then, given that $[\alpha_1, \alpha_2, \cdots, \alpha_n]$ are the zeros of *f(x)*. From here, we have the following conditions given as d_i satisfied, (where $i \in \{1,2,3, \cdots, n\}$):

$$(d_1). = \sum_{i=0}^{n} \alpha_i = \frac{-a_{n-1}}{a_n}, (d_2). = \sum_{i\neq j}^{n} \alpha_i \alpha_j = \frac{-a_{n-2}}{a_n}, (d_3). = \sum_{i<j<k}^{n} \alpha_i \alpha_j \alpha_k = \frac{-a_{n-3}}{a_n},$$

$$(d_4). = \sum_{i<j<k<l}^{n} \alpha_i \alpha_j \alpha_k \alpha_l = \frac{a_{n-4}}{a_n}, (d_5). = \sum_{i_1<i_2<i_3<i_4<i_5}^{n} \alpha_{i_1} \alpha_{i_2} \alpha_{i_3} \alpha_{i_4} \alpha_{i_5} = \frac{-a_{n-5}}{a_n}$$

$$\cdots$$

$$(d_k). = \sum_{i_1<i_2<i_3<i_4<i_5<\cdots<i_k}^{n} \alpha_{i_1} \alpha_{i_2} \alpha_{i_3} \alpha_{i_4} \alpha_{i_5} \cdots \alpha_{i_k} = \frac{-a_{n-k}}{a_n} \cdots$$

$$\cdots (d_n) = \Pi_{k=1}^{n} \alpha_{i_k} = \frac{(-1)^k a_0}{a_n}$$

5. Results

The proofs for the statement of problem.

By induction, we have as follows:

(a) The trivial case

lf $n = 0$. We have $f(x) = a_0 = C,$ a constant polynomial. This case is trivial.□

(b) The linear case

For the linear case, lf $n = 1$.

We have, $f(x) = a_1 x + a_0 \Rightarrow x = \alpha_1 = \frac{-a_0}{a_1}$, satisfying d_1.□

(c) The quadratic case

We have

$$f(x) = a_2 x^2 + a_1 x + a_0 = 0 \tag{5}$$

By theorem (2), f has at most two zeros. Let α_1 and α_2 be the zeros and let them be represented by α and β respectively . Then, we have:

$F(x) = [(x-\alpha)(x-\beta)] = 0 \Rightarrow [x^2 - (\alpha+\beta)x + \alpha\beta] = 0$. Dividing (5) by a_2, we have ,

$$f_1(x) = x^2 + \frac{a_1}{a_2}x + \frac{a_0}{a_2} = 0 \tag{6}$$

$$f_2(x) = x^2 - (\alpha+\beta)x + \alpha\beta = 0 \tag{7}$$

Comparing (6) and (7), we have that $\alpha + \beta = \frac{-a_1}{a_2}, \alpha\beta = \frac{a_0}{a_2}$. This satisfies (d_1), and (d_2) as stated in the foregoing conditions □

(d) The cubic case, $n = 3$

Let $\alpha, \beta,$ and γ be the solutions . This implies from theorem (2)) that

$F(x) = [(x-\alpha)(x-\beta)(x-\gamma),] \Rightarrow F(x) = [x^3 + (\alpha+\beta+\gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma] = 0.$

But $F(x) = [x^3 + \frac{a_2}{a_3}x^2 + \frac{a_1}{a_3}x + \frac{a_0}{a_3}.]$ This actually, is in line with $(d_1), (d_2)$ and (d_3).

And we have that [$(\alpha+\beta+\gamma)] = \frac{-a_2}{a_3}, [(\alpha\beta + \alpha\gamma + \beta\gamma)] = [\frac{a_1}{a_3}$ and $\alpha\beta\gamma] = [\frac{a_0}{a_3}]$ □

(e) The quartic case, $n = 4$

Set

$$F(x) = x^4 + \frac{a_3}{a_4}x + \frac{a_2}{a_4}x^2 + \frac{a_1}{a_4}x + \frac{a_0}{a_4} = 0 \tag{8}$$

Let α, β, γ and δ be the zeros fo f. We have that :

$F(x) = [(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)] = 0$

$F(x) = [x^4 - (\alpha + \beta + -\gamma + \delta)x^3 + (\alpha\beta + -\alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)x^2 - (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)x + \alpha\beta\gamma\delta] = 0$ (9)

Comparing (8) and (9), $\Rightarrow [\alpha+\beta+\gamma+\delta] = \frac{-a_3}{a_4}, [(\alpha\beta + -\alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)] = \frac{a_2}{a_4}, (\alpha\beta\gamma + \alpha\beta\gamma\delta + \alpha\gamma\delta + \beta\gamma\delta) = \frac{a_1}{a_4}$, and $\alpha\beta\gamma\delta = \frac{a_0}{a_4}$ □

(f) The quintic case, $n = 5$

Proposition 6. Let

$$f(x) = \sum_{i=0}^{5} a_i x^i.$$

Then, the conditions $d_1, d_2, \cdots, d_5$ are satisfied.

Proof: By theorem (2), there exist five zeros (roots) $\alpha_i, i = 1, \cdots, 5$ such that $f(\alpha_i) = 0$. We have : $f(x) = a^5(x^5 + \frac{a_4}{a^5}x^4 + \frac{a_3}{a_5}x^3 + \frac{a_2}{a^5}x^2 + \frac{a_1}{a_5}x + \frac{a_0}{a^5}) = 0 \Rightarrow [a^5(x-\alpha_i)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)(x-\alpha_5)] = 0. \Rightarrow$

$$[(x-\alpha_i)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)(x-\alpha_5)] = 0 \tag{10}$$

Expanding (10), we have that

$$x^5 - (\sum_{i=1}^{5} \alpha_i)x^4 + (\sum_{i\neq j}^{5} \alpha_i\alpha_j)x^3 - (\sum_{i\neq j\neq k}^{5} \alpha_i\alpha_j\alpha_k)x^2 + (\sum_{i<j<k<l}^{5} \alpha_i\alpha_j\alpha_k\alpha_l)x - \Pi_{i=1}^{5}\alpha_i$$

By this, it is observed that

$$\sum_{i=1}^{5} \alpha_i = -\frac{a_4}{a_5}, \sum_{i\neq j}^{5} \alpha_i\alpha_j = \frac{a_3}{a_5}, \sum_{i\neq j\neq k}^{5} \alpha_i\alpha_j\alpha_k = \frac{a+2}{a_5},$$

$$\sum_{i<j<k<ll}^{5} \alpha_i\alpha_j\alpha_k\alpha_l = \frac{a_1}{a_5}, and \quad \Pi_{i=1}^{5}\alpha_i \frac{a_0}{a_5}$$

□

And so, for a polynomial of degree n, we have as follows :

Proposition 7. Suppose that

$$f(x) = \sum_{i=0}^{n} a_i x^i.$$

Then, thre exist n zeros, gven by : $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ such that the conditions $d_1, d_2, \cdots, d_n$ are satisfied.

Proof: Following the forgoing theorem as given above ,, we can deduce that

$$F(x) = a^n \Pi_{i=1}^{n}(x-\alpha_i) = 0$$

$$\Rightarrow f(x) = x^n - (\sum_{i=1}^{n} \alpha_i)x^{n-1} + (\sum_{i1<i_2}^{n} \alpha_{i_1}\alpha_{i_2})x^{n-2} - (\sum_{i_1<i_2<i_3}^{n-2} \alpha_{i_1}\alpha_{i_2}\alpha_{i_3})x^{n-3}$$

$$+ \cdots + (-1)^k(\sum \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}x^{n-k} + \cdots + (-1)^n\Pi_{k=1}^{n}\alpha_{i_k}$$

Hence, this has completely satisfied the axioms $d_1, d_2, \cdots d_n$.□

6. Conclusions

The existence of the zeros depends on the degree of the polynomial and the stated given conditions.

7. Open problem

We now leave it open to our esteemed readers to also take the consideration of the zeros of polynomials in two indeterminates as stated in equation (4) and also to extend the results to the polynomials of degree *n*.

Acknowledgements. The author would like to use this particular medium to appreciate and show gratefulness unreservedly to the anonymous reviewers for their helpful comments, suggestions and other forms of contributions which have thus improved the overall quality of the work considerably.

Conflict of interest. The authors declare that they have no conflict of interest.

Authors' Contributions. Single author paper. The author did all the work for the paper.

REFERENCES

1. D.R.Wilkins, *Introduction To Galois Theory*, David R. Wilkins (2006).
2. John Kennedy, *Some Polynomial Theorems Mathematics Department Santa Monica College*, Pico Blvd. Santa Monica, CA 90405, 1900.
3. Igor Semaev and AndreaTenti, Probabilistic analysis on Macaulay matrices over finite fields and complexity of constructing Grbner bases, *Journal of Algebra*, (2021) 651 – 674.
4. F.S.Macaulay, *The Algebraic Theory of Modular Systems*, Cambridge University Press, 1916.
5. Sarah Nakato, *Non-Absolutely Irreducible ElementElements in the Ring of Integer - Valued Polynomia* (2020).
6. E.Artin, *Theory: Lectures Delivered at the University of Notre Dame* (Notre Dame Mathematical Lectures, Number 2). Dover, Mineola, NY, 1997.
7. H.M.Edwards, *Galois Theory*, Springer-Verlag, New York, 1984.
8. J.B.Fraleigh, *A First Course in Abstract Algebra*, 7th ed. Pearson, Upper Saddle River, NJ, 2003.
9. L.Gaal, *Classical Galois Theory with Examples*, American Mathematical Society, Providence, 1979.
10. D.J.H.Garling, *A Course in Galois Theory*, Cambridge University Press, Cambridge, 1986.
11. I.Kaplansky, *Fields and Rings*, 2^{nd} ed. University of Chicago Press, Chicago, 1972.
12. T.Rothman, *The Short Life of variste Galois*, Scientific American, (1982) 136 – 149.
13. W.Thomas, Judson Robert and A. Beezer, *Abstract Algebra.Theory and Applications*, Stephen F. Austin State University of Puget Sound (2015).
14. Aderemi O. Kuku, *Abstract Algebra*, Ibadan University Press (1992).
15. M.M.Rahman et al., Numerical solutions of volterra integral equations of second kind with the help of Chebyshev polynomials, *Annals of Pure and Applied Mathematics*, 1 (2) (2012) 158-167.