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Some Properties of Extended Gamma and Beta Matrix Functions involving 3-Parameter Mittage-Leffler Matrix Function

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Abstract. In this article, we involve three- parameter Mittage -Leffler matrix function for the study of the extended Gamma and Beta matrix functions. Mittage-Leffler matrix function requires for computing the context of Fractional Calculus theory. In this context, we also investigate symmetric relations, integral representations, summation relations, generating relations, and functional relations of these extended matrix functions.

Keywords: Mittage-Leffler function, Gamma matrix function, Beta matrix function, Matrix Fractional Calculus.

AMS Mathematics Subject Classification (2010): 33B15

1. Introduction

The theory of generalized special matrix functions has witnessed a rather significant evolution during the last two decades. The reasons of interest have manifold motivations. Special functions of a matrix argument appear in the study of spherical functions on certain symmetric spaces and multivariate analysis in statistics [1]. In the framework of orthogonal matrix polynomials, Beta functions of two matrix arguments has been recently used in [2], and in [3] for the case where one of the two matrix arguments is a scalar multiple of the identity. Lots of researchers [4-8] were influenced to work in the field of special functions with matrix arguments. To discuss our article we require some basic knowledge of previous special matrix functions.

Let P and Q be two positive stable matrices in $\mathbb{C}^{N\times N}$. The Gamma matrix function $\Gamma(P)$ and Beta matrix function $B(P,Q)$ have been studied in [9], as follows

$$\Gamma(P) = \int_0^\infty e^{-t}\, t^{P-1} dt\,; \qquad t^{p-1} = exp\big((P-I)\ln t\big)\,, \tag{1}$$

and

$$\mathrm{B}(P,Q) = \int_0^1 t^{P-1}\,(1-t)^{Q-1} dt\,. \tag{2}$$

Let P and Q be commuting matrices in $\mathbb{C}^{N\times N}$ such that the matrices $P+nI, Q+nI$ and $P+Q+nI$ are invertible for every integer $n\geq 0$, then according to [10], we have

$$\mathrm{B}(P,Q)=\Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q) \tag{3}$$

Let P be a positive stable matrices in $\mathbb{C}^{N\times N}$ and x is a positive real number then the incomplete gamma matrix function $\gamma(P,x)$ and its complement $\Gamma(P,x)$ are defined in [11,12] by

$$\gamma(P,x)=\int_0^x e^{-t}t^{P-1}\,dt \tag{4}$$

and

$$\Gamma(P,x)=\int_x^\infty e^{-t}\,t^{P-1}dt, \tag{5}$$

respectively, satisfy the following decomposition formula

$$\Gamma(P)=\ \gamma(P,x)+\Gamma(P,x). \tag{6}$$

Generalized Gamma and Beta matrix functions has been defined in [5] and [13], as follows

$$\Gamma_R(A)=\int_0^\infty t^{A-1}\,e^{-\left(tI+\frac{R}{t}\right)}dt;\quad t^{A-I}=exp\big((A-I)\ln t\big) \tag{7}$$

where A and R be two positive stable matrices in $\mathbb{C}^{N\times N}$.
And

$$\mathrm{B}(P,Q;R)=\int_0^1 t^{p-I}(1-t)^{Q-I}\,exp\left(\frac{-R}{t(1-t)}\right)\,dt \tag{8}$$

where P,Q and R are positive stable matrices in $\mathbb{C}^{N\times N}$.

Remark 1. If $R=O$, then $\Gamma_O(A)=\Gamma(A)$ and $\mathrm{B}(P,Q;O)=\mathrm{B}(P,Q)$ given in equations (1) and (2).

Two-parameter Mittage-Leffler matrix function studied by Grrappa and Popolizio [6] in the following form:

$$E_{(\alpha,\gamma)}(\mathrm{B})=\sum_{n=0}^{\infty}\frac{\mathrm{B}^n}{\Gamma(\alpha n+\gamma)} \tag{9}$$

where B is positive stable matrices in $\mathbb{C}^{N\times N}$ and $R(\alpha),R(\gamma)>0$.

Now we introduced a new three-parameter Mittage-Leffler matrix function in the following form:

$$E_{(\alpha,\gamma)}^{(\delta)}(\mathrm{B})=\sum_{n=0}^{\infty}\frac{\Gamma(n+\delta)\,\mathrm{B}^n}{\Gamma(\delta)\,\Gamma(\alpha n+\gamma)\,n!} \tag{10}$$

where B is positive stable matrices in $\mathbb{C}^{N\times N}$ and $R(\alpha),R(\gamma)\ and\ R(\delta)>0$.

The article is organized as follows. In section 2 we introduce new extensions of the Gamma and Beta matrix function by using a three–parameter Mittage-Leffler matrix function. Section 3 deals with the symmetric, summation, functional and generating relations of the new extended Beta matrix function. In Section 4 we investigate integral representations. Finally, some concluding remarks are collected in section 5.

2. Main results

In this part, we introduce new extensions of Gamma and Beta matrix functions by using three-parameter Mittage-Leffler matrix function.

Definition 2.1. The new extension of Gamma matrix function is given by

$$\Gamma_{(\delta)}^{(\alpha,\gamma)}(A,R) = \int_0^\infty t^{A-I}\, E_{(\alpha,\gamma)}^{(\delta)}\left(-tI - \frac{R}{t}\right) dt, \tag{11}$$

where A and R be two positive stable matrices in $\mathbb{C}^{N\times N}$ and $R(\alpha), R(\gamma)\ and\ R(\delta) > 0$.

Remark 2.1

1. If $\delta = 1$,in equation (11) then $E_{(\alpha,\gamma)}^{(\delta)}(\mathrm{B}) = E_{(\alpha,\gamma)}^{(1)}(\mathrm{B})$ and $\Gamma_{(\delta)}^{(\alpha,\gamma)}(A,R) = \Gamma_{(1)}^{(\alpha,\gamma)}(A,R)$ reduce to extended Beta matrix function presented by Goyal [14]
2. $If\ \alpha = \gamma = \delta = 1,$
$in\ equation\ (11)\ we\ obtain\ extended\ Gamma\ matrix\ function\ (7)\ as$
$$\Gamma_{(1)}^{(1,1)}(A,R) = \Gamma_R(A). \tag{12}$$
3. If we set $\alpha = \gamma = \delta = 1$ and $R = O$, in equation (11) we obtain gamma function (1)
$$\Gamma_{(1)}^{(1,1)}(A,O) = \Gamma(A) \tag{13}$$

Definition 2.2. The new extension of Beta matrix function is given by

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \int_0^1 t^{P-I}(1-t)^{Q-I} E_{(\alpha,\gamma)}^{(\delta)}\left(\frac{-R}{t(1-t)}\right) dt, \tag{14}$$

where P, Q and R are positive stable matrices in $\mathbb{C}^{N\times N}$ and $R(\alpha), R(\gamma)\ and\ R(\delta) > 0$.

Remark 2.2.

1. If $\delta = 1$, in equation (14) then $E_{(\alpha,\gamma)}^{(\delta)}(\mathrm{B}) = E_{(\alpha,\gamma)}^{(1)}(\mathrm{B})$ and $\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \mathrm{B}_{(1)}^{(\alpha,\gamma)}(P,Q;R)$ reduce to extended Beta matrix function presented by Goyal [14].
2. If $\alpha = \gamma = \delta = 1$, in equation (14) we obtain extended Beta matrix function (8) as
$$\mathrm{B}_{(1)}^{(1,1)}(P,Q;R) = \mathrm{B}(P,Q;R) \tag{15}$$
3. If we set $\alpha = \gamma = \delta = 1$ and $R = O$, in equation (14) we obtain Beta function (2) as
$$\mathrm{B}_{(1)}^{(1,1)}(P,Q;O) = \mathrm{B}(P,Q) \tag{16}$$

3. Symmetric, summation, functional and generating relations

Theorem 3.1. (Symmetric relation) Let $R(\alpha), R(\gamma),\ R(\delta) > 0\ \ and\ P, Q$ and R are positive stable matrices in $\mathbb{C}^{N\times N}$ such that $PQ = QP$, then the new extended functional relation is given by:

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(Q,P;R) \tag{17}$$

Proof: On substituting $t = 1-u$ in equation (14) we obtain

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \int_0^1 u^{Q-I}(1-u)^{P-I} E_{(\alpha,\gamma)}^{(\delta)}\left(\frac{-R}{u(1-u)}\right) du \tag{18}$$

By using equation (14), we get desired result

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(Q,P;R).$$

Theorem 3.2. (Summation relation) Let $R(\alpha), R(\gamma), R(\delta) > 0$ *and* P, Q and R are positive stable matrices in $\mathbb{C}^{N\times N}$, the new extended Beta matrix function satisfies the summation relation

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \sum_{n=0}^{\infty} \mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P+nI, Q+nI; R) \tag{19}$$

Proof: Rewrite the equation (14), we have

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \int_0^1 t^{P-I}(1-t)^Q\,(1-t)^{-I}\, E_{(\alpha,\gamma)}^{(\delta)}\left(\frac{-R}{t(1-t)}\right) dt. \tag{20}$$

Using $(1-t)^{-I} = \sum_{n=0}^{\infty} t^{nI}$ (matrix identity) in equation (20), we get

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \int_0^1 t^{P-I}(1-t)^Q \left(\sum_{n=0}^{\infty} t^{nI}\right) E_{(\alpha,\gamma)}^{(\delta)}\left(\frac{-R}{t(1-t)}\right) dt \tag{21}$$

Re-arranging the terms after changing the order of integration and summation, we have

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \sum_{n=0}^{\infty} \int_0^1 t^{P+nI-I}(1-t)^Q\, E_{(\alpha,\gamma)}^{(\delta)}\left(\frac{-R}{t(1-t)}\right) dt \tag{22}$$

By using definition of (14) in (22), we obtained desired result.

Theorem 3.3. The another summation relation for the positive stable matrices P, Q, R and $I-Q$ in $\mathbb{C}^{N\times N}$, is given by:

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P, I-Q; R) = \sum_{n=0}^{\infty} \frac{(Q)_n}{n!} \mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P+nI, I; R), \tag{23}$$

Provided $R(\alpha), R(\gamma)$ *and* $R(\delta) > 0.$

Proof: By definition of (14), we have

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P, I-Q; R) = \int_0^1 t^{P-I}(1-t)^{-Q}\, E_{(\alpha,\gamma)}^{(\delta)}\left(\frac{-R}{t(1-t)}\right) dt. \tag{24}$$

Using the matrix identity $(1-t)^{-Q} = \sum_{n=0}^{\infty} \frac{(Q)_n}{n!} t^{nI}$ in equation (24), we have

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P, I-Q; R) = \int_0^1 t^{P-I}\left(\sum_{n=0}^{\infty} \frac{(Q)_n}{n!} t^{nI}\right) E_{(\alpha,\gamma)}^{(\delta)}\left(\frac{-R}{t(1-t)}\right) dt \tag{25}$$

Re-arranging the terms after changing the order of integration and summation, we get

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P, I-Q; R) = \sum_{n=0}^{\infty} \frac{(Q)_n}{n!} \int_0^1 t^{P+nI-I}\, E_{(\alpha,\gamma)}^{(\delta)}\left(\frac{-R}{t(1-t)}\right) \tag{26}$$

Using definition (14), we get the required result.

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P, I-Q; R) = \sum_{n=0}^{\infty} \frac{(Q)_n}{n!} \mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P+nI, I, R). \tag{27}$$

Theorem 3.4. (Functional relation) Let $R(\alpha), R(\gamma), R(\delta) > 0$ *and* P, Q and R are positive stable matrices in $\mathbb{C}^{N\times N}$ then new extended functional relation is given by:

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P, Q+I; R) + \mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P+I, Q; R) \tag{28}$$

Proof. $\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P, Q+I; R) + \mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P+I, Q; R)$

$$= \int_0^1 t^{P-I}(1-t)^Q\, E_{(\alpha,\gamma)}^{(\delta)}\left(\frac{-R}{t(1-t)}\right) dt + \int_0^1 t^P(1-t)^{Q-I}\, E_{(\alpha,\gamma)}^{(\delta)}\left(\frac{-R}{t(1-t)}\right) dt \tag{29}$$

$$= \int_0^1 [t^{-1} + (1-t)^{-1}] t^P (1-t)^Q\, E_{(\alpha,\gamma)}^{(\delta)}\left(\frac{-R}{t(1-t)}\right) dt \tag{30}$$

$$= \int_0^1 t^{P-I}(1-t)^{Q-I}\, E_{(\alpha,\gamma)}^{(\delta)}\left(\frac{-R}{t(1-t)}\right) dt. \tag{31}$$

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P, Q+I; R) + \mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P+I, Q; R). \tag{32}$$

Theorem 3.5. (Generating relation) Let P, Q and R are positive stable matrices in $\mathbb{C}^{N\times N}$ and $R(\alpha), R(\gamma), R(\delta) > 0$ then the extended generating relation is given by:

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \sum_{n=0}^{\infty} \frac{(-R)^n \Gamma(n+\delta)}{\Gamma(\alpha n+\gamma)\Gamma(\delta)\, n!}\, \mathrm{B}(P-nI, Q-nI), \qquad (33)$$

Proof: From equation (14), we have:

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \int_0^1 t^{P-I}(1-t)^{Q-I}\, E_{(\alpha,\gamma)}^{(\delta)}\left(\frac{-R}{t(1-t)}\right) dt, \qquad (34)$$

Using definition (10), we obtain:

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \int_0^1 t^{P-I}(1-t)^{Q-I}\left(\sum_{n=0}^{\infty} \frac{(-R)^n \Gamma(n+\delta)}{t^n(1-t)^n \Gamma(\alpha n+\gamma)\Gamma(\delta) n!}\right) dt. \qquad (35)$$

On interchanging the order of integration and summation, we get

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \sum_{n=0}^{\infty} \frac{(-R)^n \Gamma(n+\delta)}{\Gamma(\alpha n+\gamma)\Gamma(\delta) n!} \int_0^1 t^{P-nI-I}(1-t)^{Q-nI-I} dt. \qquad (36)$$

Using definition (2), we obtain required result.

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \sum_{n=0}^{\infty} \frac{(-R)^n \Gamma(n+\delta)}{\Gamma(\alpha n+\gamma)\Gamma(\delta)\, n!}\, \mathrm{B}(P-nI, Q-nI). \qquad (37)$$

4. Integral representations

In this section, we derive integral representations for the new extended Beta matrix function in the form of the following theorems:

Theorem 4.1. Let P, Q and R are positive stable matrices in $\mathbb{C}^{N\times N}$ and $R(\alpha), R(\gamma), R(\delta) > 0$ then the following integral formula holds true for the new extended Beta matrix function:

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = 2\int_0^{\frac{\pi}{2}} (cos(\lambda))^{2P-1} (sin(\lambda))^{2Q-1}\, E_{(\alpha,\gamma)}^{(\delta)}\left(-R\, sec^2(\lambda) csc^2(\lambda)\right) d\lambda\,, \qquad (38)$$

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = 2\int_0^{\frac{\pi}{2}} (sin(\lambda))^{2P-1} (cos(\lambda))^{2Q-1}\, E_{(\alpha,\gamma)}^{(\delta)}\left(-R\, sec^2(\lambda) csc^2(\lambda)\right) d\lambda\,, \qquad 39)$$

$$\mathrm{B}_{(\delta)}^{(\alpha,\gamma)}(P,Q;R) = \int_0^{\infty} \frac{u^{P-1}}{(1+u)^{P+Q}}\, E_{(\alpha,\gamma)}^{(\delta)}\left(-2R - R\left(u+\frac{1}{u}\right)\right) du, \qquad (40)$$

Proof: In definition (14), putting $t = cos^2(\lambda)$ and $t = sin^2(\lambda)$ and $= \frac{u}{1+u}$, we obtained the results (38),(39) and (40) respectively.

5. Concluding remarks and future works

In the presented work we have introduced three- parameter Mittage-Leffler matrix function and involved it to introduced new extended Gamma and Beta functions. We have investigated some properties of these extended matrix functions. Application of special functions to matrix function theory are not limited, many extensions of special matrix functions and polynomials have been obtained by authors in different literatures. In the future, we need to do more work on special matrix functions some of these points are as follows

(1) Study the applications of special matrix functions and polynomials in engineering physics and biology.
(2) Develop the numerical methods for special matrix functions and polynomials.
(3) Using the special matrix functions in software application designing.

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REFERENCES

1. A.T.James, Special functions of matrix and single argument in Statistics, In theory and Applications of Special functions, (Edited by R.A.Askey), pp.497-520, Academic Press, (1975).
2. L.Jódar, R.Company and E.Ponsoda, Orthogonal matrix polynomials and systems of second order differential equations, *Diff. Equations Dynam. Syst.*, 3(3) (1995) 269-288.
3. A.G.Constantine and R.J.Muirhead, Partial differential equations for hypergeometric functions of two arguments matrix, *J. Multivariate Anal.*, 3 (1972) 332-338.
4. B.Çekim, Generalized Euler's beta matrix function and related functions. AIP Conf. Proc. 2013 (1558) 1132–1135.
5. Abdalla, M.; Bakhet, A. Extension of Beta matrix function, *Asian J. Math. Comput. Res.*, 2016, 9, 253–264.
6. R.Garrappa and M.Popolizio, Computing the matrix Mittag-Leffler function with applications to fractional calculus, *J. Sci. Comput.*, 77 (2018) 129–153.
7. M.Abdalla, Special matrix functions: Characteristics, achievements and future directions, *Linear Multilinear Algebra*, 68 (2020) 1–28.
8. A.Altin, B.Çekim and R.Sahin, On the matrix versions of Appell hypergeometric functions, *Quaest. Math.*, 37 (2014) 31–38.
9. L.Jódar and J.C.Cortés, Some properties of Gamma and Beta matrix functions. *Appl. Math. Lett.*, 11 (1998) 89–93.
10. L.Jódar and J.C.Cortés, On the hypergeometric matrix function, *J. Comput Appl. Math.*, 99 (1998) 205–217.
11. M.Abdalla, On the incomplete hypergeometric matrix functions, *Ramanujan J.*, 43 (2017) 663–678.
12. J.Sastre and L.Jódar, Asymptotics of the modified Bessel and the incomplete gamma matrix functions, *Appl Math Lett.*, 16 (2003) 815–820.
13. M.Abul-Dahab and A.A.Bakhet, Certain generalized gamma matrix functions and their properties, *J. Anal. Num. Theor.*, 3 (2015) 63–68.
14. R.Goyal, P.Agarwal, G.I.Oros and S.Jain, Extended Beta and Gamma matrix functions via 2-parmeter Mittage-Leffler matrix function, *Mathematics*, 10 (2022) 892.