

Some Characterization of Directed Below 0-distributive Join Semilattices

Rathindra Chandra Gope

Department of Computer Science and Engineerin
North East University Bangladesh,
Sylhet 3100. E-mail: rcgope@neub.edu.bd

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Abstract. Varlet characterized the concept of 0-distributive lattices to generalize the notion of pseudo complemented lattices. Powar and Thakare introduced the notion of 0-Distributive semilattices. Very few authors have studied join semilattices. Among them Nimbhorkar and Rahemani, Akhter and Noor and Rao and Kumar have introduced the join semilattices. In this paper, we have introduced the notion of join semilattices with 0 -distributive directed below. A join semilattice S is called directed below if for any $a, b \in S$ there is $d \leq a, b$ such that $d \in S$. Then obviously, every join semilattice with 0 is directed below.

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1. Introduction

The work of Noor and Momtaz Begum [1] has motivated me to make the foundation of 0 -distributive join semilattices directed below. In this paper, we have given some characterization of 0 -distributive join semilattice directed below. Rao and Kumar [2] introduced modular and classic ideals in directed below join semilattice. By [3] introduced the concept of 0 -distributive lattices and a meet semilattice S with 0 is called 0 -distributive if for all $a, b, c \in S$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge d = 0$ for some $d \geq b, c$. In section 2, I introduce some basic definitions which is needed for establish the main result of this paper.

2. Preliminaries

In this section, we consider some important relevant definitions which are already known and will be used to get the main result of this paper.

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Definition 2.1. An ordered set $\langle S; \leq \rangle$ is known as **join semilattice** if $\sup\{a, b\}$ exists for all $a, b \in S$. A join semilattice S containing an element 0 is said to be join semilattice S with 0 if $0 \leq x$ for all $x \in S$.

Definition 2.2. Then the element 0 is said to be the **least element** of S . Dually, a join semilattice S containing an element 1 is said to be join semilattice S with 1 if $x \leq 1$ for all $x \in S$. Then the element 1 is said to be the **largest element** of S .

Definition 2.3. By [6], a join semilattice S with 0 is said to be 0 -*distributive* if for all $a, b, c \in S$, $(a] \cap (b] = (a] \cap (c] = (0] \Rightarrow (x] \cap (b \vee c] = (0]$.

Definition 2.4. A join semilattice S is called **directed below** if for any $a, b \in S$, there is $d \leq a, b$ such that $d \in S$. Clearly, every join semilattice with 0 is directed below.

Definition 2.5. An upset F is called **filter** if for any $a, b \in F$, there exists $c \leq a, b$ such that $c \in F$. A non-empty subset I of join semilattice S is called **down set** if $a \in I$ and $b \leq a$ for any $b \in S$ implies that $a \in I$. A down set I of S is called an **ideal** if for all $a, b \in I$ such that $a \vee b \in I$.

Definition 2.6. A filter P (up set) is called a **prime filter** if $a \vee b \in P$ implies either $a \in P$ or $b \in P$. An ideal J of a join semilattice S is called **prime ideal** if $S - J$ is a prime filter.

3. Main result

To derive the main consequences of this paper, we need to demonstrate the following Lemmas and Theorems.

Definition 3.1. An ideal I of S is called **maximal ideal** if $S \neq I$ and it is not contained by any other proper ideal of S . For $a \in F$, the filter $F = \{b \in S \mid a \leq b\}$ is called the **principal filter** generated by a and it is denoted by $[a)$. Let S be a join semilattice. A filter F of S is called a **maximal filter** of S if $F \subseteq M \subset S$ for any filter M of S implies $M = F$.

Lemma 3.1. Suppose that F be a filter and I be an ideal of a directed below join semilattice S , such that $F \cap I = \emptyset$. In that case, F is a maximal filter disjoint from I if and only if for each $a \notin F$, there exists $b \in F$ such that $(a] \cap (b] \in I$.

Definition 3.2. Let S be a join semilattice with 0 . Let A be a sub join semilattice. Then we can define $A^\circ = \{x \in S \mid (x] \cap (a] = (0] \text{ for some } a \in A\}$ is a **semi prime ideal** of S if

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and only if S is 0-distributive and this is obviously a down set. Now if $A \subseteq B$, this implies $A^\circ \subseteq B^\circ$ and for any $a \in S$ implies $(a)^\perp = (a)^\circ = (a)^\circ$.

Lemma 3.2. Let S be a directed below join semilattice with 0. If S is not 0-distributive, the set

$$F := \{x \in S \mid x \leq (a) \cap (x) \neq 0 \text{ for all } y \leq b, c\}$$

Corollary 3.3. Let S be a directed below join semilattice with 0. Then the following conditions are equivalent.

- (i) S is 0-distributive.
- (ii) For each $a \in S$, $(a)^\perp = (a)^\circ = (a)^\circ$ is an ideal.
- (iii) Every maximal filter of S is a minimal prime ideals.

Since in a 0-distributive join semilattice S , for each $a \in S$, $(a)^\perp$ is an ideal, we can denote it by $(a)^\circ$. Talukder and Gope [6] has generalized the concept of 0-distributive join semilattices.

Now, we have the following characterization due to [1, 9].

Theorem 3.1. For any join sub semilattice A of a directed below join semilattice S with 0. Then A° is a semi prime ideal of S if and only if S is 0-distributive.

Proof: Let S is a distributive join semilattice with 0. It is obvious that A° is a down set. Thus for some $x, y \in A^\circ$, $(x) \cap (a) = (0) = (y) \cap (b)$ for some $a, b \in A$. This implies $(x) \cap (a \cap b) = (0) = (y) \cap (a \cap b)$ and since S is a distributive join semilattice with 0, then $(a \cap b) \cap d = (0)$ for some $d \geq x, y$. Now $(a \cap b) \in A$ implies $d \in A^\circ$ and then A° is an ideal. For some $x, y, z \in A^\circ$, $(x) \cap (y) \in A^\circ$ and $(x) \cap (z) \in A^\circ$. This implies $(x) \cap (y) \cap (a_1) = (0) = (x) \cap (z) \cap (b_1)$ for some $a_1, b_1 \in A$.

Then $(x) \cap (y) \cap (a_1 \cap b_1) = (0) = (x) \cap (z) \cap (a_1 \cap b_1)$, thus by the property of 0 distributive join semilattice, $(x) \cap (d_1) \in A^\circ$ as $(a_1 \cap b_1) \in A$. Hence A° is a semi prime.

Conversely, if A° is a semi prime ideal for every join sub semilattice A of S , by Lemma 3.3 it is clear that S is 0-distributive.

Definition 3.3. A prime ideal P of S is called a **minimal prime ideal** of S if $M \subseteq P$ for any prime ideal M of S implies $P = M$.

Now we have the following useful characterizations of minimal prime ideals of directed below join semilattice.

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Lemma 3.5. Suppose A and B be filters of a directed below join semilattice S with 0 , such that $A \cap B^\circ = \varphi$. Then there exists a minimal prime ideal containing B° and disjoint from A .

Proof: Let $0 \notin A \vee B$. For if $0 \in A \vee B$. Then $(a] \cap (b] \leq (0]$ for some $a \in A, b \in B$. That is, $(a] \cap (b] = (0]$, which implies $a \in B^\circ$ gives a contradiction. It follows that $A \vee B$ is a proper filter of S . Then by Lemma 3.2, $A \vee B \subseteq M$ for some maximal filter M . If $x \in M \cap B^\circ$, then $x \in M$ and $(x] \cap (b_1] = (0]$ for some $b_1 \in B \subseteq M$. This implies $0 \in M$ which is a contradiction as M is maximal. Thus $M \cap B^\circ = \varphi$. Then by definition, $S - M$ is a minimal prime ideal set containing B° . Moreover $(S - M) \cap A = \varphi$.

Lemma 3.6. Let A be filter of a directed below join semilattice S with 0 . Then A° is the intersection of all the minimal prime ideal disjoints from A .

Proof: Let N be any minimal prime ideal disjoint from A . If $x \in A^\circ$, then $(x] \cap (a] = (0]$ for some $a \in A$ and so $x \in N$ as N is prime.

Conversely, let $y \in S - A^\circ$. Then $(y] \cap (a] \neq (0]$ for all $a \in A$. Hence $A \vee (y]$ is a proper filter of S . Then by definition, $A \vee (y] \subseteq M$ for some maximal filter M . Thus $S - M$ is a minimal prime ideal. Clearly $y \notin S - M$.

Theorem 3.2. Let S be a directed below join semilattice. Then the union of any two filters of S is also a filter.

Proof: Let F and Q are two filters of a directed below join semilattice S . Let $x \in F \cup Q$ and $y \in S$ with $x \leq y$. Then $x \in F$ and $x \in Q$ implies $y \in F$ and $y \in Q$ as F and Q are both filters. Thus $y \in F \cup Q$. Now suppose that $x, y \in F \cup Q$. This implies $x, y \in F$ and $x, y \in Q$. Since F and Q are two filters, then there exists $f \in F$ and $q \in Q$ such that $x, y \leq f, q$. Let $c = f \vee q$. This implies $c \in F \cup Q$ and thus $x, y \leq c$. Hence $F \cup Q$ is filter.

Theorem 3.3. Let S be a join semilattice with 0 . Then every prime down set contains a minimal prime downset.

Proof: Let F be a prime down set of a join semilattice S with 0 and let A be a non-empty set of all prime down set Q contained in F as $F \in A$. Let M be a non-empty prime down set as $0 \in M$ and $M = \bigcup \{P \mid P \in C\}$, where $C \in A$. Let $a \in M$ as $b \leq a$. This implies $a \in P$ for all $P \in C$. Thus $b \in P$ as P is an down set. Then clearly, $b \in M$. Again let for some $x, y \in S$, $x \vee y \in M$. This implies $x \vee y \in P$ for all $P \in C$. Since P is a prime down set, this implies either $x \in P$ or $y \in P$ and this is obvious that either $x \in M$ or $y \in M$. Thus every prime down set contains a minimal prime up set.

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The following Lemma is a special characterization of directed below 0-distributive join semilattices.

Lemma 3.9. Let S be a directed below join semilattice with 0 . Then the following are equivalent.

- (i) S is 0-distributive.
- (ii) For any three filters A, B, C of L

$$((A \cap B) \vee (A \cap C))^\circ = A^\circ \cap (B \vee C)^\circ$$
- (iii) For any two filters A, B of S , $(A \cap B)^\circ = A^\circ \cap B^\circ$
- (iv) For all $a, b, c \in S$, $(a)^\circ \cap (b)^\circ = (d)^\circ$ for some $d \geq b, c$
- (v) For all $a, b, c \in S$, $(a]^\bullet \cap (b]^\bullet = (d]^\bullet$ for some $d \geq b, c$

Proof: (i) \Rightarrow (ii). Let S is 0-distributive. Since $(A \cap B) \vee (A \cap C) \subseteq A$ and $B \vee C$, so $((A \cap B) \vee (A \cap C))^\circ \subseteq A^\circ \cap (B \vee C)^\circ$. Now suppose $x \in A^\circ \cap (B \vee C)^\circ$. Then $x \in A^\circ$ and $x \in (B \vee C)^\circ$. Thus $(x] \cap (a] = (0]$ for some $x \in A$ and $(x \cap d] = (0]$ for some $d \in B \vee C$. Now $d \in B \vee C$ implies $d \in (b \cap c]$ for some $b \in B, c \in C$. Hence $(x \cap a] = (0] = (x \cap b] \cap (c]$. Then $(x \cap c] \cap (a] = (0] = (x \cap c] \cap (b]$. Since S is 0-distributive, so $(x \cap c] \cap (d_1] = (0]$ for some $d_1 \geq a, b$. Then $d_1 \in A \cap B$. Now $(x \cap a] = (0]$ implies $(x \cap d_1] \cap (a] = (0] = (x \cap d_1] \cap (c]$. Again by the 0-distributivity, $(x \cap d_1] \cap (d_2] = (0]$ for some $d_2 \geq a, c$ that is $d_2 \geq A \cap C$. Therefore, $x \in ((A \cap B) \vee (A \cap C))^\circ$ and hence (ii) holds.

(ii) \Rightarrow (iii) is trivial by considering $B = C$ in (iii).

(iii) \Rightarrow (iv). Choose $A = [a]$ and $B = [a]$ in (iii). Now for all $d \geq a, b$, $[a] \supseteq [d]$ and $[b] \supseteq [d]$ and so $[d]^\circ \subseteq (a)^\circ \cap (b)^\circ$. Also by (iii), $(a)^\circ \cap (b)^\circ = ([a] \cap [b])^\circ$. Thus, $x \in (a)^\circ \cap (b)^\circ$, implies $(x \cap d_1] = (0]$ for some $d_1 \geq a, b$. That is, $x \in (d_1)^\circ$ for some $d_1 \geq a, b$. Thus (iv) holds.

(ii) \Leftrightarrow (iv) is obvious.

(v) \Rightarrow (i). Suppose (v) holds and for $a, b, c \in S$, $(a \cap b] = (0] = (a \cap c]$. Then $a \in (b]^\bullet \cap (c]^\bullet = (d]^\bullet$ for some $d \geq b, c$. Therefore, $(a \cap d] = (0]$ and so S is 0-distributive.

4. Conclusion

In this study, some characterization of directed below 0-distributive join semilattices are established.

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