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Split Feasibility Problem and Fixed Point Problem for Asymptotically Strictly Pseudo Nonspreading Mapping

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Abstract. The purpose of this paper is to introduce an iterative algorithm for finding a common element of the solution set of split feasibility problem and the fixed point set of a asymptotically strictly pseudo nonspreading mapping in the Hilbert Space.

Keywords: Split feasibility, Asymptotically strictly pseudo nonspreading mapping, Hilbert space

AMS Mathematics Subject Classification (2010): 47J20, 49J40, 49J52

1. Introduction

The Split feasibility problem was originally introduced by Censor and Elfving [7] for modeling phase retrieval problems and it later was studied extensively as an extremely powerful tool for the treatment of a wide range of inverse problem, such as medical image reconstruction and intensity modulated radiation therapy problems. For example we may refer to [8-10]. Let C and Q be two nonempty closed convex subset of real Hilbert spaces H_1 and H_2 respectively and $A : H_1 \to H_2$ be a bounded linear operator. The Split Feasibility Problem (SFP) is to find a point x such that

$$x \in C, Ax \in Q \tag{1.1}$$

Throughout the paper, we denote by Γ, the solution set of the split feasibility problem that is

$$\Gamma = \{x \in C : Ax \in Q = C \cap (A^{-1}Q)\}$$

Finding the common solution of a Split Feasibility Problem and fixed point problem is one of the core interest of many researchers. Recently Ceng et. al. [5] introduced a relaxed extragradient method with regularization for finding a common solution set of SFP and the set of $Fix(T)$ of the fixed point of nonexpansive mapping T. Recently Deepho [6] introduced and analyzed a relaxed extragradient method with regularization for finding a common element of the solution set Γ of the Split Feasibility Problem and fixed point set Fix of an uniformly Lipschitz continuous and asymptotically quasi nonexpansive mapping in the setting of real Hilbert space. Very recently Ansari et al. [2] deals with the weak convergence of the relaxed extragradient method with regularization for computing a common element of the solution set of Split Feasibility Problem and fixed point set of asymptotically k strict pseudo contractive mapping in intermediate sense.

2. Some definitions

Definition 2.1. [4] Let H be real Hilbert Space and C be a non empty closed convex subset of H, a mapping $T: C \to C$ is said to be nonspreading if

$$2||x-y||^2 \leq ||Tx-y||^2 + ||Ty-x||^2 \ \forall\, x,y \in C$$

the above inequality is equivalent to

$$||x-y||^2 \leq ||x-y||^2 + 2 < x-Tx, y-Ty > \forall\, x,y \in C$$

Definition 2.2. [4] Let H be real Hilbert Space. A mapping $T : D(T) \subset H \to H$ is said to be k strict pseudo nonspreading mapping if there exist $k \in [0,1)$ such that

$$||Tx-Ty||^2 \leq ||x-y||^2 + k||x-Tx-(y-Ty)||^2 + 2 < x-Tx, y-Ty > \quad \forall\, x,y \in D(T)$$

Every nonspreading mapping is k strict pseudo nonspreading mapping.

Definition 2.3. [4] Let H be real Hilbert Space. A mapping $T: D(T) \subset H \to H$ is said to be k asymptotically strictly pseudo nonspreading mapping if there exists $k \in [0,1)$ and a sequence $k_n \subset [1,\infty)$ with $k_n \to 1 (n \to \infty)$ such that

$$||T^n x - T^n y||^2 \leq k_n||x-y||^2 + k||x-T^n x-(y-T^n y)||^2 + 2 < x-T^n x, y-T^n y > \forall\, x,y \in D(T)$$

It is easy to see that the class of k asymptotically strictly pseudo nonspreading mapping is more general than the class of k strictly pseudo nonspreading mapping and k asymptotically strictly pseudo contraction mapping.

3. Preliminaries

Let H be a real Hilbert Space whose inner product and norm are denoted by $<.,.>$ and $||.||$ respectively. We denote the strong convergence and weak convergence of a sequence $\{x_n\}$ to a point $x \in X$ by $x_n \to x$ and $x_n \rightharpoonup x$ respectively. Let K be a nonempty closed convex subset of real Hilbert Space H for every point $x \in H$, there exist a unique nearest point of K, denoted by $P_K x$, such that $\|x - P_K x\| = \|x-y\| \ \forall\, x,y \in K$. Such a P_K is called the metric projection from H onto K. It is well known that P_K is firmly nonexpansive mapping from H onto C, i.e.,

$$||P_K x - P_K y||^2 \leq < P_K x - P_K y, x-y > \forall\, x,y \in H$$

Proposition 3.1 [2]. For a given $x \in H$ and $z \in K$, we have

(1) $z = P_K x$ if and only if $< x-z, z-y \geq 0 \quad \forall\, y \in K$;

(2) $z = P_K x$ if and only if $||x-z||^2 \leq ||x-y||^2 - ||y-z||^2 \quad \forall\, y \in K$;

(3) $< P_K x - P_K y > \geq ||P_K x - P_K y||^2 \ \forall\, y \in K$.

Let K be a non empty closed and convex subset of H and $F : K \to H$ be a mapping. The Variational inequality problem (VIP) is to find $x \epsilon K$ such that

$$< Fx, y - x > \geq 0 \;\forall\; y \epsilon K \tag{3.1}$$

the solution of VIP is denoted by $VI(K, F)$. It is well known that

$$x\epsilon\, VI(K, F) \Leftrightarrow x = P_K(x - \lambda Fx) \;\forall\; \lambda > 0$$

A set valued mapping $T : H \to 2^H$ called monotone if $< x - y, f - g > \geq 0$ whenever $f \in Tx$, $g \in Ty$. It is said to be maximal monotone if, in addition, the graph $G(T) = \{(x, f) \in H \times H : f \in Tx\}$ of T is not properly contained in the graph of any other monotone operator. It is well known that a monotone mapping T is maximal if and only if, for

$(x, f) \in H \times H, < x - y, f - g > \geq 0$ for every $(y, g) \in G(T) \Rightarrow f \in Tx$

Let $F : K \to H$ be a monotone that is $< Fx - Fy, x - y > \geq 0$ $for\ all\ x, y \in K$ and k lipschitz continuous mapping, let $N_k v$ be the normal cone to K at $v \in K$, that is

$$N_k\, v = \{\omega\epsilon\, H : < v - u, w > \geq 0\ for\ all\ u\epsilon\, K\}$$

Define

$$Tv = \begin{cases} Fv + N_k\, v & , if\ v\epsilon\, K \\ \varphi & , \quad otherwise \end{cases}$$

Then T is maximal monotone set valued mapping. It is well known that if $0 \epsilon Tv$ then $-Fv\epsilon\, N_k v$, which is further equivalent to the variational inequality.

Proposition 3.2. [5] Let C and Q be nonempty closed subsets of Hilbert space H_1 and H_2 respectively and $A : H_1 \to H_2$ be a bounded linear operator. For given $x^* \epsilon H_1$, the following statement are equivalent

(1) x^* solves the SFP;

(2) x^* solves the Fixed point equation $P_c(I - \lambda A^*(I - P_Q)A)x^* = x^*$;

(3) x^* solves the VIP of finding $x^* \epsilon C$ such that $< \nabla f(x^*), x - x^* > \geq 0$ for all $x \epsilon C$ where $\nabla f = A^*(I - P_Q)A$ and A^* is the adjoint of A.

Lemma 3.3. [2] Let H be real Hilbert space.Then for all $x, y\epsilon\, H$ we have

- $\|x - y\|^2 \leq \|x\|^2 + \|y\|^2$
- $\|\lambda x - (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$, for all $\lambda \epsilon [0,1]$

Lemma 3.4. [4] Let K be a non empty closed convex subset of a real Hilbert space H and let $T : K \to K$ be a continuous k asymptotically strictly pseudo nonspreading mapping if $F(T) \neq \phi$, then it is a closed and convex subset.

Lemma 3.5. [4] Let K be a non empty closed convex subset of a real Hilbert space H and let $T : K \to K$ be a continuous k asymptotically strictly pseudo nonspreading mapping then $(I - T)$ is demiclosed at 0 that is , if $x_n \rightharpoonup x^*$and $limsup_{m\to\infty}\, limsup_{n\to\infty}\, ||(I - T^m)x_n\| = 0$ then $||(I - T)x^*\| = 0$.

Lemma 3.6. [3] Let K be a non empty closed convex subset of a real Hilbert space H and let $T : K \to K$ be a continuous k asymptotically strictly pseudo nonspreading mapping and uniformly L Lipschitzian mapping then for any sequence $x_n$in K converging weakly to a point p and $\{\|x_n - Tx_n\|\}$ converging strongly to 0, we have $p = Tp$.

Motivated by the above, the purpose of this paper is to is to introduce an iterative algorithm for finding a common element of the solution set of split feasibility problem and the fixed point set of a asymptotically strictly pseudo nonspreading mapping in the Hilbert Space which improve and extends the results of [1].

4. Main result

Theorem 4.1. Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be an uniformly L lipschitzian and k asymptotically strictly pseudo nonspreading mapping such that $Fix(T) \cap \Gamma \neq \phi$.

$$x_0 \epsilon H$$
$$y_n = P_C(I - \lambda_n \nabla f_{(\alpha_n)}(\mathrm{x_n}))$$
$$x_{n+1} = (1 - \beta_n)x_n + \beta_n T^n(y_n)$$

Assume that the sequence $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$, $\{k_n\}$ satisfy the following conditions,
(1) $\sum_{n=1}^{\infty} \alpha_n < \infty$;
(2) $\{\lambda_n\} \subset [a, b]$ for some $a, b \;\epsilon \left(0, \frac{1}{||A||^2}\right)$, $\sum_{n=1}^{\infty} \lambda_n < \infty$;
(3) $\{\nabla f_{(\alpha_n)}(x_n)\}_{n=1}^{\infty}$ is bounded sequence;
(4) $\{\beta_n\} \subset [d, e]$ for some $d, e \;\epsilon (0, 1)$;
(5) $\{k_n\} \subset [1, \infty)$, $k \;\epsilon [0, 1)$.
Then both the sequence $\{x_n\}$ and $\{y_n\}$ converges weakly to an element$x^* \epsilon\, Fix(T) \cap \Gamma$.
Proof. Let $p\epsilon\, Fix(T) \cap \Gamma$ be arbitrary chosen, then we have $T(p) = p \;\epsilon\, C$ and $Ap \in Q$. Therefore,

$$P_C(p) = p \text{ and } P_Q(Ap) = Ap$$

Since P_C is nonexpansive, we have

$$\|y_n - p\|^2 = \|P_C\left(I - \lambda_n \nabla f_{(\alpha_n)}(\mathrm{x_n})\right) - P_C(p)\|^2$$
$$\leq \|(x_n - P) - \lambda_n \nabla f_{(\alpha_n)}(\mathrm{x_n})\|^2$$

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 + \lambda_n^2 \left\|\nabla f_{(\alpha_n)}(\mathrm{x_n})\right\|^2 \tag{4.1}$$

Since $y_n \epsilon\, C$ and $T^n y_n \epsilon\, C$, we have

$$\|y_n - T^n y_n\|^2 = \|P_C\left(I - \lambda_n \nabla f_{(\alpha_n)}(\mathrm{x_n})\right) - P_C(T^n y_n)\|^2$$
$$\leq \|(x_n - T^n y_n) - \lambda_n \nabla f_{(\alpha_n)}(\mathrm{x_n})\|^2$$

$$\|y_n - T^n y_n\|^2 \leq \|(x_n - T^n y_n\| + \lambda_n^2 \|\nabla f_{(\alpha_n)}(\mathrm{x_n})\|^2 \tag{4.2}$$

By k asymptotically strictly pseudo nonspreading mapping of T, by Lemma (3.3), (4.1) and (.4.2)

$\|x_{n+1}-p\|^2 = \|(1-\beta_n)x_n + \beta_n T^n(y_n) - p\|^2$
$= \|x_n - \beta_n x_n + \beta_n T^n(y_n) - \beta_n p + \beta_n p - p\|^2$
$= \|x_n - \beta_n x_n) + \beta_n(T^n(y_n) - p) + (\beta_n p - p\|^2$
$= \|(1-\beta_n)x_n + \beta_n(T^n(y_n) - p) + (\beta_n - 1)p\|^2$
$= \|(1-\beta_n)(x_n - p) + \beta_n(T^n(y_n) - p)\|^2$
$= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|T^n(y_n) - p\|^2 - \beta_n(1-\beta_n)\|x_n - p - T^n y - p\|^2$
$= (1-\beta_n)||x_n - p||^2 + \beta_n||T^n(y_n) - p||^2 - \beta_n(1-\beta_n)||x_{n+1} - p||^2||T^n y - x_n||^2$
$= (1-\beta_n)||x_n - p||^2 + \beta_n\left\{k_n||y_n - p|| + k||y_n - T^n y_n||^2\right\}||(T^n y_n - p)||^2 -$
$(1-\beta_n)||T^n y - x_n||^2$
$= (1-\beta_n)||x_n - p||^2 + \beta_n\left\{k_n\{||x_n - p||^2 + |\lambda_n|^2\left||\nabla f_{(\alpha_n)}(x_n)\right||^2\}\right.$
$\left. + k||y_n - T^n y_n||^2\right\} - \beta_n(1-\beta_n)||T^n y - x_n||^2$
$=(1-\beta_n)||x_n - p||^2 + \beta_n k_n||x_n - p||^2 + \beta_n k_n \lambda_n^2\left||\nabla f_{(\alpha_n)}(x_n)\right||^{\mathbf{2}} + \beta_n k||y_n -$
$T^n y_n||^2 - \beta_n(1-\beta_n)||T^n y - x_n||^2$
$= (1-\beta_n+\beta_n k_n)||x_n - p||^2 + \beta_n k_n \lambda_n^2\left||\nabla f_{(\alpha_n)}(x_n)\right||^{\mathbf{2}} + \beta_n k||y_n - T^n y_n||^2$
$- \beta_n(1-\beta_n)||T^n y - x_n||^2$
$= (1-\beta_n+\beta_n k_n)||x_n - p||^2 + \beta_n k_n \lambda_n^2\left||\nabla f_{(\alpha_n)}(x_n)\right||^{\mathbf{2}} + \beta_n k(||x_n - T^n y_n||^2$
$+ |\lambda_n|^2\left||\nabla f_{(\alpha_n)}(x_n)\right||^2) - \beta_n(1-\beta_n)||T^n y - x_n||^2$
$= (1-\beta_n+\beta_n k_n)||x_n - p||^2 + \beta_n k_n \lambda_n^2\left||\nabla f_{(\alpha_n)}(x_n)\right||^{\mathbf{2}} + \beta_n k||x_n - T^n y_n||^2$
$+ |\lambda_n|^2\beta_n k\left||\nabla f_{(\alpha_n)}(x_n)\right||^2 - \beta_n(1-\beta_n)||T^n y - x_n||^2$
$= (1-\beta_n+\beta_n k_n)||x_n - p||^2 + (|\lambda_n|^2\beta_n k + \beta_n k_n \lambda_n^2)\left||\nabla f_{(\alpha_n)}(x_n)\right||^{\mathbf{2}}$
$+\beta_n k||x_n - T^n y_n||^2 - \beta_n(1-\beta_n)||T^n y - x_n||^2$
$= (1+\beta_n(k_n - 1))||x_n - p||^2 + (|\lambda_n|^2\beta_n(k+k_n)M$
$+||x_n - T^n y_n||^2(\beta_n k - \beta_n(1-\beta_n))$

$$||x_{n+1} - p||^2 \le (1+\beta_n(k_n - 1))||x_n - p||^2 + (|\lambda_n|^2\beta_n(k+k_n)M + ||x_n - T^n y_n||^2(\beta_n k - \beta_n(1-\beta_n)) \quad (4.3)$$

$$||x_{n+1} - p||^2 \le (1+\beta_n(k_n - 1))||x_n - p||^2 + b_n$$

$\sum_{n=1}^{\infty}\beta_n(k_n-1)<\infty$ and $\sum_{n=1}^{\infty}\lambda_n<\infty, 0<k<1$. We have $\sum_{n=1}^{\infty}b_n<\infty$ and $lim_{n\to\infty}\left||x_n-p|\right|$ exists. Also, $lim_{n\to\infty}\left||y_n-p|\right|\, exists.$

Thus from (4.3), we obtain

$$\left||x_{n+1}-p|\right|^2\leq(1+\beta_n(k_n-1))\left||x_n-p|\right|^2+(|\lambda_n|^2\beta_n(k+k_n)M$$
$$+\left||x_n-T^n y_n|\right|^2(\beta_n k-\beta_n(1-\beta_n))$$

$$\left||x_n-T^n y_n|\right|^2(\beta_n k-\beta_n(1-\beta_n))\leq(1+\beta_n(k_n-1))\left||x_n-p|\right|^2+(|\lambda_n|^2\beta_n(k+k_n)M-\left||x_{n+1}-p|\right|^2$$

Using limiting conditions of M and k, we get,

$$lim_{n\to\infty}\left||T^n y_n-x_n|\right|=0 \tag{4.4}$$

$$lim_{n\to\infty}\left||T^n y_n-y_n|\right|=0 \tag{4.5}$$

By (4.4),

$$\left||x_{n+1}-x_n|\right| = \left||(1-\beta_n)x_n+\beta_n T^n(y_n)-x_n|\right|$$
$$=\left||(x_n-\beta_n x_n)+\beta_n T^n(y_n)-x_n|\right|^2$$
$$=\left||(x_n-\beta_n x_n)+\beta_n T^n(y_n)-x_n|\right|^2$$
$$=lim_{n\to\infty}\beta_n\left||T^n y_n-x_n|\right|\to 0\,.$$
$$\left||x_{n+1}-x_n|\right|\to 0 \tag{4.6}$$

Since $y_n=P_C\left(x_n-\lambda_n\nabla f_{(\alpha_n)}(\mathrm{x_n})\right)$ and by proposition 3.1, we have

$$\left||y_n-p|\right|^2\leq\left||x_n-\lambda_n\nabla f_{(\alpha_n)}(\mathrm{x_n})-p\right||^2-\left||x_n-\lambda_n\nabla f_{(\alpha_n)}(\mathrm{x_n})-y_n|\right|$$
$$=\left||x_n-p|\right|^2-\left||x_n-y_n|\right|^2+2\lambda_n<\nabla f_{(\alpha_n)}(\mathrm{x_n}),p-y_n>$$
$$=\left||x_n-p|\right|^2-\left||x_n-y_n|\right|^2+2\lambda_n<\nabla f_{(\alpha_n)}(\mathrm{x_n}),p-x_n>+2\lambda_n$$
$$<\nabla f_{(\alpha_n)}(\mathrm{x_n}),x_n-y_n>$$
$$=\left||x_n-p|\right|^2-\left||x_n-y_n|\right|^2+2\lambda_n<\nabla f_{(\alpha_n)}(\mathrm{x_n})-\nabla f_{(\alpha_n)}(\mathrm{p}),p-x_n>+2\lambda_n$$
$$<\nabla f_{(\alpha_n)}(p),p-x_n>+2\lambda_n<\nabla f_{(\alpha_n)}(\mathrm{x_n}),x_n-y_n>$$
$$=\left||x_n-p|\right|^2-\left||x_n-y_n|\right|^2+2\lambda_n<\nabla f_{(\alpha_n)}(\mathrm{p}),p-x_n>+2\lambda_n$$
$$<\nabla f_{(\alpha_n)}(\mathrm{x_n}),x_n-y_n-p+p>$$
$$=\left||x_n-p|\right|^2-\left||x_n-y_n|\right|^2+2\lambda_n<\nabla f(\mathrm{p})+(\alpha_n)p,p-x_n>+2\lambda_n$$
$$<\nabla f_{(\alpha_n)}(\mathrm{x_n}),x_n-p>+<2\lambda_n<\nabla f_{(\alpha_n)}(\mathrm{x_n})+y_n-p>$$

$$=\left||x_n-p|\right|^2-\left||x_n-y_n|\right|^2+2\lambda_n<\nabla f(p),p-x_n>+2\lambda_n<\alpha_n p,p-x_n>+2\lambda_n$$
$$<\nabla f_{(\alpha_n)}(\mathrm{x_n}),x_n-p>+2\lambda_n<\nabla f_{(\alpha_n)}(\mathrm{x_n}),y_n-p>$$

$$= ||x_n - p||^2 - ||x_n - y_n||^2 + 2\lambda_n < \alpha_n p, p - x_n > +2\lambda_n < \nabla f_{(\alpha_n)}(\mathrm{x_n}), x_n - p > +2\lambda_n < \nabla f_{(\alpha_n)}(\mathrm{x_n}), y_n - p >$$

$$||y_n - p||^2 \leq ||x_n - p||^2 - ||x_n - y_n||^2 + 2\lambda_n\alpha_n||p||||p - x_n|| + 2\lambda_n \left||\nabla f_{(\alpha_n)}(\mathrm{x_n})|\right| ||x_n - p|| + 2\lambda_n \left||\nabla f_{(\alpha_n)}(\mathrm{x_n})|\right| ||y_n - p|| \qquad (4.7)$$

Now using Lemma (3.3) and (4.4),

$$||x_{n+1} - x_n||^2 = ||(1 - \beta_n)x_n + \beta_n T^n(y_n) - p||^2$$

$$||(x_n) - (\beta_n)x_n + \beta_n T^n(y_n) - p + (\beta_n p) - (\beta_n p)||^2$$
$$= ||(1 - \beta_n)x_n + \beta_n(T^n y_n - p) - p(1 - \beta_n)||^2$$
$$= ||(1 - \beta_n)(x_n - p) + \beta_n(T^n y_n - p)||^2$$
$$= ||(1 - \beta_n)(x_n - p) + \beta_n(T^n y_n - p)||^2 - \beta_n(1 - \beta_n)||T^n y_n - x_n||^2$$
$$= (1 - \beta_n)||x_n - p||^2 + \beta_n\{k_n||y_n - p||^2 + k||y_n - T^n y_n||^2 - \beta_n(1 - \beta_n)||T^n y_n - x_n||^2$$
$$= (1 - \beta_n)||x_n - p||^2 + \beta_n k_n||y_n - p||^2 + \beta_n k||y_n - T^n y_n||^2 - \beta_n(1 - \beta_n)||T^n y_n - x_n||^2$$
$$= (1 - \beta_n)||x_n - p||^2 + \beta_n k_n\{||x_n - p||^2 - ||x_n - y_n||^2 + 2\lambda_n\alpha_n||p||||p - x_n|| + 2\lambda_n| \left|\nabla f_{(\alpha_n)}(\mathrm{x_n})\right| |||x_n - p|| + 2\lambda_n \left||\nabla f_{(\alpha_n)}(\mathrm{x_n})|\right| ||y_n - p||^2\} + \beta_n k||y_n - T^n y_n||^2 - \beta_n(1 - \beta_n)||T^n y_n - x_n||^2$$

Taking limit both sides and using conditions , we obtain

$$lim_{n\to\infty}||x_n - y_n|| = 0$$

Now,

$$||y_{n+1} - y_n|| = ||P_C\left(I - \lambda_{n+1}\nabla f_{(\alpha_{n+1})}(\mathrm{x_{n+1}})\right) - P_C\left(I - \lambda_n\nabla f_{(\alpha_n)}(\mathrm{x_n})\right)|$$
$$\leq \left(x_{n+1} - \lambda_{n+1}\nabla f_{(\alpha_{n+1})}(\mathrm{x_{n+1}})\right) - \left(x_n - \lambda_n\nabla f_{(\alpha_n)}(\mathrm{x_n})\right)$$
$$\leq \left(||x_{n+1} - x_n| + \lambda_{n+1}\left|\left|\nabla f_{(\alpha_{n+1})}(\mathrm{x_{n+1}})\right||^2\right) + \left(\lambda_n\left|\left|\nabla f_{(\alpha_n)}(\mathrm{x_n})\right|\right|^2\right)$$

Taking limit both the sides

$$\lim_{n\to\infty} ||y_{n+1} - y_n|| = 0 \qquad (4.8)$$

Since $||y_{n+1} - y_n|| \to 0$, $\|T^n y_n - y_n\| \to 0$ as $n \to \infty$. T is uniformly Lipschitz by Lemma (3.6), $||Ty_n - y_n|| \to 0$ as $n\to\infty$. Since $\{x_n\}$ is bounded sequence, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$that converges weakly to x^* i.e., $x_n \rightharpoonup x^*$.
Let $\{x_{n_j}\}$ of $\{x_n\}$ such that converges weakly to x^* i.e., $x_{n_j} \rightharpoonup x^*$.

Assume $x' \neq x^*$. By Opial's condition

$$\lim_{n\to\infty} ||x_n - x^*|| = 0$$

$$\liminf_{n\to\infty} ||x_{n_i} - x^*|| < \lim_{n\to\infty} ||x_{n_i} - x'||$$
$$\liminf_{n\to\infty} ||x_{n_i} - x'|| = \lim_{n\to\infty} ||x_{n_i} - x'||$$
$$< \lim_{j\to\infty} ||x_{n_j} - x^*|| = \lim_{n\to\infty} ||x_n - x^*||$$

This contradict to our assumption $x' \neq x^*$.
Hence $x_{n_j} \rightharpoonup x^*, x_n \rightharpoonup x^*$. For all $f \in H, f(x_n) \to f(x^*)$
Next we show that $y_n \rightharpoonup x^*$

$$\left||f(y_n) - f(x^*)|\right| = \left||f(y_n) + f(x_n) - f(x_n) - f(x^*)|\right|$$
$$\leq \left||f(y_n) - f(x_n)|| + ||f(x_n) - f(x^*)|\right|$$

$\lim_{n\to\infty} \left||f(y_n) - f(x^*)|\right| = 0$ for all $f \in H, f(x_n) \to f(x^*)$, $y_n \rightharpoonup x^*$

By lemma 3.5, $x^* \in Fix(T)$.

Now we show that $x^* \in \Gamma$

$$S\omega_1 = \begin{cases} \lambda_n \nabla f_{\omega_1} + N_C\omega_1 & if\ \omega_1 \in C \\ \emptyset & otherwise \end{cases}$$

$N_C\omega_1 = \{z \in H: < \omega_1 - u, z > \geq 0$ for all $u \in C$

To show that $x^* \in \Gamma$ it is sufficient to show that $0 \in Sx^*$
Let $(\omega_1, z) \in G(C)$.
We have, $z \in S\omega_1 - \lambda_n \nabla f_{\omega_1} + N_C\omega_1$
And, $z - \lambda_n \nabla f_{\omega_1} \in N_C\omega_1$
So we have $< \omega_1 - u, z - \lambda_n \nabla f_{\omega_1} > \geq 0$ for all $u \in C$
Since $\omega_1 \in C$
We have $y_n = P_C(I - \lambda_n \nabla f_{(\alpha_n)})(\mathrm{x_n})$

And by Proposition 3.1,
$< (x_n - \lambda_n \nabla f_{(\alpha_n)}(\mathrm{x_n}) - y_n, y_n - \omega_1 > \geq 0$
$< \omega_1 - y_n, y_n - x_n + \lambda_n \nabla f_{(\alpha_n)}(\mathrm{x_n}) > \geq 0$

$z_n + \lambda_n \nabla f_{\omega_1} \in N_C\omega_1$ and $y_{n_i} \in$ C. It follows that
$< \omega_1 - y_{n_i}, z > \geq < \omega_1 - y_{n_i}, \nabla f_{\omega_1} >$
$\geq < \omega_1 - y_{n_i}, \lambda_{n_i} \nabla f_{\omega_1} > - < \omega_1 - y_{n_i}, y_{n_i} - x_{n_i} + \lambda_{n_i} \nabla f_{(\alpha_{n_i})} x_{n_i} >$
$\geq \langle \omega_1 - y_{n_i}, \lambda_{n_i} \nabla f_{\omega_1} > - < \omega_1 - y_{n_i}, y_{n_i} - x_{n_i} + \lambda_{n_i} \nabla \mathrm{f} x_{n_i} > - \lambda_{n_i}(\alpha_{n_i})$
$< \omega_1 - y_{n_i}, \lambda_{n_i} \rangle$

$=\langle \omega_1 - y_{n_i}, \lambda_{n_i}\nabla f_{\omega_1} - \lambda_{n_i}\nabla f y_{n_i} > - < \omega_1 - y_{n_i}, \lambda_{n_i}\nabla f x_{n_i} > - < \omega_1 - y_{n_i}, y_{n_i} - x_{n_i} >$
$-\lambda_{n_i}(\alpha_{n_i)} < \omega_1 - y_{n_i}, \lambda_{n_i}\rangle$
$\geq \langle \omega_1 - y_{n_i}, \lambda_{n_i}\nabla f_{\omega_1} - \lambda_{n_i}\nabla f_{x_{n_i}} > - < \omega_1 - y_{n_i}, y_{n_i} - x_{n_i} > -\lambda_{n_i}(\alpha_{n_i)}$
$< \omega_1 - y_{n_i}, x_{n_i}\rangle$

Taking limit as $i \to \infty$, we obtain $< \omega_1 - x^*, z > \geq 0$ as $i \to \infty$

Since $< \omega_1 - x^*, z - 0 > \geq 0$ for every $(\omega_1, z) \in G(S)$.
Therefore the maximality of S implies that $0 \in Sx^*$.
Thus we have, $x^* \in VI(C, \nabla f)$ finally, proposition [5] implies that $x^* \in \Gamma$. This completes the proof.

Remark 2.6. Theorem [10] improve and extends [1] in the following aspects:

1. The technique of proving weak convergence in [10] is different from that in [1] because of our technique to use k asymptotically strictly pseudo nonspreading mapping and the property of maximal monotone mappings.
2. The problem of finding a common element of $Fix(T \cap \Gamma)$ for k asymptotically strictly pseudo nonspreading mappings which is more general than that for nonexpansive mappings and the problem of finding a solution of the SFP in [1].
3. The problem of finding a common element of $Fix(T \cap \Gamma)$ for k asymptotically k strictly pseudo nonspreading mappings which is more general than that for asymptotically k strict pseudo contractive mappings and the problem of finding a solution of the SFP in [2].

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