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On the Diophantine Equation $p^x + (2p-1)^y = z^2$ where p and $2p-1$ are Primes

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Abstract. This study presents proof and solutions of all non-negative integrals of the Diophantine equation $p^x + (2p-1)^y = z^2$ when p and $2p-1$ are primes. This proof uses the prediction of Catalan's conjecture and related theories in proving by separating $p = 2, p = 3$ and $p > 3$. It shows that if $p = 2$, the Diophantine equation $p^x + (2p-1)^y = z^2$ is in the form of $2^x + 3^y = z^2$ which has three solutions; $(0,1,2), (3,0,3)$ and $(4,2,5)$ and if $p = 3$ then $(1,0,2)$ is the solutions of equation $3^x + 5^y = z^2$ when considering the value of $p > 3$, the Diophantine equation $p^x + (2p-1)^y = z^2$ has no solutions.

Keywords: Diophantine equations, exponential equations

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

Many studies claim that the Diophantine equation is one of the classic problems in elementary number theory and algebraic number theory. In 1844, Catalan [2] proved that a conjecture $(a, b, x, y) = (3, 2, 2, 3)$ is a unique solution of the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$.

Later in 2012, Sroysang [8] proved that The Diophantine equation $3^x + 5^y = z^2$ has a unique non-negative integer solution. The solution (x, y, z) is $(1, 0, 2)$.

In 2013, Sroysang [9] studied solutions to the Diophantine equation $2^x + 3^y = z^2$. $(0,1,2)$, $(3,0,3)$ and $(4,2,5)$ are only three solutions for where x, y and z are non-negative integers. and in 2014, show that the Diophantine equation $7^x + 31^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.

In 2014, Suvarnamani [1], found that $(p, q, x, y, z) = (3, 5, 1, 0, 2)$ is a unique solution of the Diophantine equation $p^x + q^y = z^2$ where p is an odd prime number which $q - p = 2$ and x, y and z are non-negative integers.

In 2017, Burshtein [3] showed that the Diophantine equation $p^x + q^y = z^2$ has infinitely many solutions when $p = 2$ and also when p is prime.

Additionally, in 2018, Kumar et al. [6,7] showed that on the non-linear Diophantine equation $p^x + (p+6)^y = z^2$, when p and $p + 6$ both are primes with $p =$

6n+1 has no solution, where x, y, and z are non-negative integer and n is a natural number on the Non-Linear Diophantine equation, $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$

Moreover, Fernando [5] also showed that the Diophantine equation $p^x + (p+8)^y = z^2$ when $p > 3$ and $p + 8$ are primes has no solution (x, y, z) in positive integers.

In 2020, Burshtein [4] showed that the Diophantine equation $p^x + (p+5)^y = z^2$ when p is prime where $p + 5 = 2^{2u}$ has no solution (x, y, z) in positive integers.

In 2021, Vipawadee [10] showed that the Diophantine equation $p^x + (p+4^n)^y = z^2$ has no solutions, where x, y, z are non-negative integers and p >3 and $p + 4^n$ are primes.

Because of this open problem, the author is therefore interested in studying the Diophantine equation; $p^x + (2p-1)^y = z^2$ when p and $2p-1$ are primes and x, y, z are non-negative integers.

2. Preliminaries

Proposition 2.1. [1] (Catalan's conjecture) (3,2,2,3) is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers such that min $\{a, b, x, y\} > 1$.

Lemma 2.1. (2,3,3) and (3,1,2) are solution $\{p, x, z\}$ for the Diophantine equation $p^x + 1 = z^2$ where x and z are non-negative integers, and p is a positive prime number.

Proof. Let p be a positive prime number and x, z are non-negative integers.
If $x = 0$, Then $z^2 = 2$ which is impossible. then $x \geq 1$.
We consider 3 cases including $p = 2, p = 3$ and $p > 3$

Case 1. If $p = 2$, then $2^x = (z-1)(z+1)$. Thus there exist non-negative integers α, β such that $2^\alpha = z + 1$ and $p^\beta = z - 1$, where $\alpha > \beta$ and $\alpha + \beta = x$.
Therefore, $2^\beta(2^{\alpha-\beta} - 1) = 2$. This implies that $\beta = 1$ and $2^{\alpha-\beta} = 2$. Then $\alpha = 2$, then $(2,3,3)$ is a solution (p, x, z) for the Diophantine equation $p^x + 1 = z^2$.

Case 2. If $p = 3$, then it follows that $p^x = (z-1)(z+1)$.
Hence, $p^\beta(p^{\alpha-\beta} - 1) = 2$ where $\alpha > \beta$ and $\alpha + \beta = x$.
This implies that $\beta = 0$ and $p^x - 1 = 2$.
Thus, $p = 3$ and $x = 1$, which yield the solution $(p, x, z) = (3,1,2)$.

Case 3. If $p > 3$, then $p \equiv 1 (mod\ 4)$ or $p \equiv -1 \equiv 3 (mod\ 4)$.
Since z is even and $z^2 \equiv 0 (mod\ 4)$.

Subcase 1. Suppose that $p \equiv 1 (mod\ 4)$. Then $p^x + 1 \equiv 2 (mod\ 4)$ which is a contradiction since $z^2 \equiv 0 (mod\ 4)$.

Subcase 2. Suppose that $p \equiv -1 \equiv 3 (mod\ 4)$.

If $x = 2k, k \geq 1$, then $p^x + 1 = z^2 \equiv 2 (mod\ 4)$.
which is a contradiction since $z^2 \equiv 0 (mod\ 4)$.

If $x = 2k + 1$, $k \geq 0$, then $1 + p^{2k+1} = z^2$, $p^{2k+1} = (z-1)(z+1)$,
Thus there exist non-negative integers α, β such that $p^\alpha = z + 1$ and $p^\beta = z - 1$,
where $\alpha > \beta \geq 0$ and $\alpha + \beta = x = 2k + 1$.
Therefore $p^\beta(p^{\alpha-\beta} - 1) = 2$. This implies that $\beta = 0$ and $p^{2k+1} - 1 = 2$,
Then $p^{2k+1} = 3$ which is impossible.
Hence, the Diophantine equation $p^x + 1 = z^2$ has no solutions where $p > 3$.

Lemma 2.2. $(2,1,2)$ is a unique solution (p, y, z) for the Diophantine equation $1 + (2p-1)^y = z^2$ where p and $2p-1$ are primes and y, z are non-negative integers.

Proof. Suppose that $(2p-1)^y + 1 = z^2$.
then $z^2$is even which implies that $z^2 \equiv 0 (mod\ 4)$.
We consider 3 cases.

Case 1. If $y = 0$, Then $z^2 = 2$ which is impossible.

Case 2. If $y \geq 1$ and $p = 2$, Thus $z^2 = 3^y + 1 \geq 4$, that is $z \geq 2$. Hence, $\min\{p, y, z\} > 1$, then $(2,1,2)$ is a unique solution (p, x, z) for the Diophantine equation $(2p-1)^y + 1 = z^2$.

Case 3. If $y \geq 1$ and ≥ 3 , Thus $z^2 = (2p-1)^y \geq 5$, that is $z \geq 2$.
we have $(2p-1)^y = (z-1)(z+1)$.
Suppose $\alpha > \beta \geq 0$ are integers such that $\alpha + \beta = y$.
Therefore, $2 = (2p-1)^\alpha - (2p-1)^\beta = (2p-1)^\beta[(2p-1)^{\alpha-\beta} - 1]$.
This implies that $\beta = 0$ and $(2p-1)^y = 3$ which is impossible. Hence, the Diophantine equation $1 + (2p-1)^y = z^2$ has no solutions where $p \geq 3$.

3. Main result

Theorem 3.1. [8] $(0,1,2), (3,0,3)$ and $(4,2,5)$ are only three solutions for the Diophantine equation $2^x + 3^y = z^2$ where x, y and z are non-negative integers.

Theorem 3.2. [9] $(1,0,2)$ is a unique solution (x, y, z) for the Diophantine equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers.

Theorem 3.3. $(2,3,3), (3,1,2), (2,1,2), (0,1,2), (3,0,3), (4,2,5)$ and $(1,0,2)$ are only seven solutions for the Diophantine equation $p^x + (2p-1)^y = z^2$ where p and $2p-1$ are prime and x, y and z are non-negative integers.

Proof. Let p and $2p-1$ be prime, and x, y and z are non-negative integers.
Suppose that $$p^x + (2p-1)^y = z^2 \quad (1)$$
then $z^2$is even which implies that $z^2 \equiv 0 (mod\ 4)$.
We consider 3 cases including $x = 0$ and $x \geq 1$.

Case 1. If $x = 0$ and $y = 0$ Then $z^2 = 2$ which is impossible.
If $= 0, y \geq 1$, then $(2,3,3)$ and $(3,1,2)$ are solution (p, y, z) for the Diophantine equation $1 + (2p-1)^y = z^2$ where y and z are non-negative integers and p is a positive prime number (by Lemma 2.2).

Case 2. If $x \geq 1$ and $y = 0$ then $(2,1,2)$ is a unique solution (p, y, z) for the Diophantine equation $p^x + 1 = z^2$ where p is a positive prime number and x, z are non-negative integers (by Lemma 2.1).

Case 3. If $x \geq 1, y \geq 1$ From (1), then we consider three subcases including $p = 2$, $p = 3$ and $p > 3$.

Subcase 1: [9] Suppose $p = 2$, Then $(0,1,2), (3,0,3)$ and $(4,2,5)$ are only three solutions for the Diophantine equation $p^x + (2p-1)^y = z^2$ where x, y and z are non-negative integers (by Theorem 3.1).

Subcase 2: [8] Suppose $p = 3$, the Diophantine equation $p^x + (2p-1)^y = z^2$ has a unique non-negative integer solution. The solution (x, y, z) is (1,0,2) (by Theorem 3.2).

Subcase 3: Suppose $p > 3$, where $p = \pm 1 (mod\ 4)$. From (1), it follows that $z^2 \equiv 0\ \equiv (-1)^x + 1\ (mod\ 4)$, then $p \equiv -1 (mod\ 4)$ or $p = 4N + 3$ and x is odd, so we let $x = 2n + 1$ where n is non-negative integers.

We separate the subcase proofs as $y = 2m$ and $y = 2m + 1$.

(a): If $y = 2m$, where m is positive integers.

we have $p^x + (2p-1)^{2m} = z^2$ it is written as

$$p^x = (z - (2p-1)^m)(z + (2p-1)^m) \quad (2)$$

From (2), yield (3) and (4).

$$p^\alpha = z + (2p-1)^m \quad (3)$$

and $$p^\beta = z - (2p-1)^m \quad (4)$$

where $0 \leq \beta < \alpha \leq x$ and $\alpha + \beta = x = 2n + 1$.

From (3) and (4), we have $p^\beta(p^{\alpha-\beta} - 1) = 2(2p-1)^m$.

This implies that $\beta = 0$, then $p^{2n+1} - 1 = 2(2p-1)^m$

For $n = 0$, we obtain $p - 1 = 2(2p-1)^m$.

Then $$p = 2(2p-1)^m + 1 \quad (5)$$

where $p \equiv -1 (mod\ 4)$ or $p = 4N + 3$. from (5),

we have $4N + 3 = 2\big(2(4N+3)\big)^m + 1$.

Hence, $2N + 1 = \big(2(4N+3)\big)^{m,}$ which is impossible.

For $n \geq 1$, we have $p^{2n+1} - 1 = 2(2p-1)^m$

Then $(p-1)(p^{2n} + p^{2n-1} + \cdots + p + 1) = 2(2p-1)^m$

It follows that p − 1 is an even positive divisor of $2(2p-1)^m$,

That is $p - 1 = 2(2p-1)^j$, where j is an integer such that 0 ≤ j < m.

For j = 0, p = 3 which contradicts the fact that p > 3.

For 1 ≤ j < m, we obtain $2(2p-1)^j = (2p-1) - p$ or

$2(2p-1)^j + p = 2p - 1$, which is impossible.

(b): If $y = 2m + 1$, where m is positive integers.

From (1), we have $z^2 \equiv (2p-1)^y (mod\ p)$. (6)

Since $2p - 1$ and $p = 4N + 3$ are prime. Such that $p \nmid (2p-1)$.

Since $(2p-1)^y \equiv -1 \equiv z^2 (mod\ p)$

Therefore $z^2 + 1 \equiv 0 (mod\ p)$

Let ${z_1}^2$ be any solution of $z^2 + 1 \equiv 0 (mod\ p)$, so that ${z_1}^2 \equiv -1 (mod\ p)$. Because $p \nmid z_1$, from Fermat's theorem $1 \equiv {z_1}^{p-1} \equiv ({z_1}^2)^{\frac{(p-1)}{2}} \equiv (-1)^{\frac{(p-1)}{2}}\ (mod\ p)$.

The possibility that $p = 4N + 3$ for some N does not arise. If it did, we would have

$$(-1)^{(p-1)/2} = (-1)^{2N+1} = -1$$

Therefore, $1 \equiv -1 (mod\ p)$. The net result of this is that $p|2$, which is patently false.

Hence, the Diophantine equation $p^x + (2p-1)^y = z^2$ where p and $2p - 1$ are prime and x, y and z are non-negative integers. has no solution where $p > 3$.

Corollary 3.4. The Diophantine equation $p^x + (2p-1)^y = u^{2n}$ has no solution. where $p > 3, p$ and $2p - 1$ are primes and x, y and u are non-negative integers and n is a positive number.

Proof. Let $u^n = z$ where z are non-negative integers.
then $p^x + (2p-1)^y = u^{2n} = z^2$ has no solution by Theorem 3.3.

Corollary 3.5. The Diophantine equation $p^x + (2p-1)^y = u^{2n+2}$ has no solution where $p > 3$, p and $2p-1$ are prime and x, y, u are non-negative integers and n is a natural number.
Proof. Let $u^{n+1} = z$ where z are non-negative integers.
then $p^x + (2p-1)^y = u^{2n+2} = z^2$, which has no solution by Theorem 3.3.
Corollary 3.6. The Diophantine equation $p^x - (2p-1)^y = z^2$ has no solution where $p > 3$ and $p = 4N + 3$ is a positive prime and x, y, z are non-negative integers and N is a natural number.
Proof: Suppose that $p^x - (2p-1)^y = z^2$, since $p \nmid z$ and $p \nmid (2p-1)$.
Then $z^2 \equiv -(2p-1)^y \equiv -1 (mod\, p)$.
Therefore $z^2 + 1 \equiv 0 (mod\, p)$
Let b^2 be any solution of $z^2 + 1 \equiv 0 (mod\, p)$, so that $b^2 \equiv -1 (mod\, p)$. Because $p \nmid b$, from Fermat's theorem, is $1 \equiv b^{p-1} \equiv (b^2)^{\frac{(p-1)}{2}} \equiv (-1)^{\frac{(p-1)}{2}} (mod\, p)$.
The possibility that $p = 4N + 3$ for some N does not arise. If it did, we would have

$$(-1)^{(p-1)/2} = (-1)^{2N+1} = -1$$

Therefore, $1 \equiv -1 (mod\, p)$. The net result of this is that $p|2$, which is patently false.
Hence, the Diophantine equation $p^x - (2p-1)^y = z^2$ where p and $2p-1$ are prime and x, y and z are non-negative integers, has no solution when $p > 3$.

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REFERENCES

1. A.Suvarnamani, Solutions of the Diophantine equation $p^x + q^y = z^2$, *International Journal of Pure and Applied Mathematics*, 94 (4) (2014) 457-460.
2. P.Mihail, Primary cycolotomic units and a proof of Catalan's conjecture, *J. Reine. Angew. Math.*, 27 (2004) 167-195.
3. N.Burshtein, On the diophantine equation $p^x + q^y = z^2$, when $p = 2$ and $p = 3$. *Annals of Pure and Applied Mathematics*, 13(2) (2017) 229– 233.
4. N.Burshtein, On the Diophantine equation $p^x + (p+5)^y = z^2$, when $p + 5 = 2^{2u}$. *Annals of Pure and Applied Mathematics*, 18(1) (2020) 41–44.
5. N.Fernando, On the solvability of the diophantine equation $p^x + (p+8)^y = z^2$, when $p > 3$ and $p + 8$ are primes, *Annals of Pure and Applied Mathematics*, 18(1) (2018) 9–13.
6. S.Kumar, S.Gupta and H.Kishan, On the non-linear diophantine equation $p^x + (p+6)^y = z^2$, *Annals of Pure and Applied Mathematics*, 8(1) (2018) 125–128.

7. S. Kumar, S.Gupta and H.Kishan, On the non-linear diophantine equation $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$, *Annals of Pure and Applied Mathematics*, 18(1) (2018) 94-94.
8. B.Sroysang, The diophantine equation $3^x + 5^y = z^{2.}$, *International Journal of Pure and Applied Mathematics*, 81(4) (2012) 605-608.
9. B.Sroysang, More on the diophantine equation $2^x + 3^y = z^2$, *International Journal of Pure and Applied Mathematics*, 84(2) (2013) 133-137.
10. M.Vipawadee, On the Diophantine equation $p^x + (p + 4^n)^y = z^2$, *Annals of Pure and Applied Mathematics*, 23(2) (2021) 117-121.