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A Study on Degrees of all Neighbors of a Vertex in Fuzzy Graphs

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Abstract. Graphs are one of the main tools for the mathematical modelling of various human problems. Fuzzy graphs clearly showed their ability to solve uncertain and ambiguous problems. Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $u \in V$. We define $S_G^F(u) = \sum_{vu\in\mathcal{E}} \mu(vu)\deg_G v$. In this paper, we present some properties of $S_G^F(u)$ on fuzzy graphs and establish relations between the Zagreb indices and the sum of the degrees of all neighbors of a vertex.

Keywords: Fuzzy set, fuzzy graph, fuzzy common neighborhood, degrees of neighbors.

AMS Mathematics Subject Classification (2010): 05C07, 03E72, 05C72

1. Introduction

The concept of graph theory was first introduced by Euler. In 1965, Zadeh discussed the fuzzy set [40]. The first definition of a fuzzy graph was given by Kaufmann, which was based on Zadeh's fuzzy relations in [13]. However, the development of fuzzy graph theory is due to the ground-setting papers of Rosenfeld [19] and Yeh and Bang [39]. In Rosenfeld's paper, basic structural and connectivity concepts were presented, while Yeh and Bang introduced different connectivity parameters and discussed their application. Rosenfeld obtained the fuzzy analoguess of several graph-theoretic concepts like bridges, paths, cycles, trees, and connectedness. Most of the theoretical development of fuzzy graph theory is based on Rosenfeld's initial work. Mordeson studied fuzzy line graphs and developed its basic properties, in 1993 [14]. Fuzzy graph theory is finding more and more applications. Applications can be found in cluster analysis, pattern classification, database theory, social sciences, neural networks, decision analysis, group structure, portfolio management, and many other areas [15].

The main purpose of this paper is to define some concepts in fuzzy graphs and get results about them that are also true in ordinary graphs. In this section, we provide formal definitions, basic concepts, and properties of fuzzy graphs. For simplicity, we consider only undirected fuzzy graphs unless otherwise specified. Thus, the edges of the fuzzy graph are unordered pairs of vertices. First, we go through some basic definitions from [14, 17]. Rashmanlou et al. [20-30] studied different kinds of fuzzy graphs. Talebi et al. [34,35,36,37,38] introduced isomorphism on interval-valued fuzzy graphs and investigated several concepts in iinterval-valued intuitionistic fuzzy graphs. Islam et al. [6-12]

investigated wiener index, Zagreb index, and F-index on fuzzy graphs. In this paper, we present some properties of $S_G^F(u)$ on fuzzy graphs and establish relations between the Zagreb indices and sum of the degrees of all neighbors of a vertex.

2. Preliminaries

Definition 2. 1*A fuzzy subset of a non-empty set S is a map $\sigma: S \to [0,1]$ which assigns to each element x in S a degree of membership $\sigma(x)$ in $[0,1]$ such that $0 \leq \sigma(x) \leq 1$. If S represents a set, a fuzzy relation μ on S is a fuzzy subset of $S \times S$. In symbols, $\mu: S \times S \to [0,1]$ such that $0 \leq \mu(x,y) \leq 1$ for all $(x,y) \in S \times S$.*

Definition 2.2. *Let σ be a fuzzy subset of a set S and μ a fuzzy relation on S. Then μ is called a fuzzy relation on σ if $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in S$ where $\wedge$ denote minimum.*

Let V be a nonempty set. Define the relation $\sim$ on $\mathrm{V} \times \mathrm{V}$ by for all $(\mathrm{x},\mathrm{y}), (\mathrm{u},\mathrm{v}) \in \mathrm{V} \times \mathrm{V}$, $(\mathrm{x},\mathrm{y}) \sim (\mathrm{u},\mathrm{v})$ if and only if $\mathrm{x} = \mathrm{u}$ and $\mathrm{y} = \mathrm{v}$ or $\mathrm{x} = \mathrm{v}$ and $\mathrm{y} = \mathrm{u}$. Then it is easily shown that $\sim$ is an equivalence relation on $\mathrm{V} \times \mathrm{V}$. For all $\mathrm{x}, \mathrm{y} \in \mathrm{V}$, let $[(\mathrm{x},\mathrm{y})]$ denote the equivalence class of (x,y) with respect to $\sim$. Then $[(\mathrm{x},\mathrm{y})] = \{(\mathrm{x},\mathrm{y}), (\mathrm{y},\mathrm{x})\}$. Let $\mathcal{E}_{\mathrm{V}} = \{[(\mathrm{x},\mathrm{y})] | \mathrm{x}, \mathrm{y} \in \mathrm{V}, \mathrm{x} \neq \mathrm{y}\}$. For simplicity, we often write $\mathcal{E}$ for $\mathcal{E}_{\mathrm{V}}$ when V is understood. Let $\mathrm{E} \subseteq \mathcal{E}$. A graph is a pair (V,E). The elements of V are thought of as vertices of the graph and the elements of E as the edges. For $\mathrm{x}, \mathrm{y} \in \mathrm{V}$, we let xy denote $[(\mathrm{x},\mathrm{y})]$. Then clearly $\mathrm{xy} = \mathrm{yx}$. We note that graph (V,E) has no loops or parallel edges.

Definition 2.3. *A fuzzy graph $G = (V, \sigma_G, \mu_G)$ is a triple consisting of a nonempty set V together with a pair of functions $\sigma := \sigma_G: V \to [0,1]$ and $\mu := \mu_G: \mathcal{E} \to [0,1]$ such that for all $x, y \in V$, $\mu(xy) \leq \sigma(x) \wedge \sigma(y)$.*
The fuzzy set σ is called the fuzzy vertex set of G and μ the fuzzy edge set of G. Clearly, μ is a fuzzy relation on σ.

Definition 2.4. *A path P in a fuzzy graph $G = (V, \sigma, \mu)$ is a sequence of distinct vertices $x_0, x_1, \cdots, x_n$ (except possibly x_0 and x_n) such that $\mu(x_{i-1}x_i) > 0$ for $i = 1, \cdots, n$. Here n is called the length of the path. We call P a cycle if $x_0 = x_n$ and $n \geq 3$. Two vertices that are joined by a path are called connected.*

Definition 2.5. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The degree $x \in V$ is denoted by $deg_G(x)$ and defined as $deg_G(x) = \sum_{y \in V} \mu(xy)$.*

Definition 2.6. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The size of G is denoted by $S(G)$ and defined as $\sum_{xy \in \mathcal{E}} \mu(xy)$.*

Definition 2.7. *The fuzzy common neighborhood graph or briefly fuzzy congraph of $G = (V, \sigma, \mu)$ is a fuzzy graph as $con(G) = (V, \omega, \lambda)$ such that $\omega(x) = \sigma(x)$ and $\lambda(uv) = min_{x \in H}\{\mu(ux). \mu(vx)\}$, where $H = N_G(u) \cap N_G(v)$.*

Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $V = \{v_1, v_2, \ldots, v_p\}$, $\mathcal{E} = \{e_1, e_2, \ldots, e_q\}$ the

vertex set and the edge set of G, *respectively.*
The adjacency matrix of fuzzy graph G *is the* $p \times p$ *matrix* $A_F = A_F(G)$ *whose* (i,j) *entry denoted by* a_{ij}, *is defined by* $a_{ij} = \mu(v_i v_j)$.

Definition 2.8. *Let* $A = [a_{ij}]$ *and* $B = [b_{ij}]$ *be two matrix of size* $m \times n$. *Then we define* $C = A \odot B$ *is the* $m \times n$ *matrix whose* (i,j) *entry denoted by* $a_{ij} \times b_{ij}$.

Two old and much studied degree–based graph invariants are the so-called first and second Zagreb indices, defined as

$$M_1(G) = \sum_{v \in V(G)} deg(v)^2 \quad and \quad M_2(G) = \sum_{uv \in E(G)} deg(u)\, deg(v) \ .$$

Also, in [16], the authors defined two new indices $N_1(G)$ *and* $N_2(G)$ *as follows.*

$$N_1(G) = \sum_{u \in V(G)} deg_G(u)\, deg_{con(G)}(u) \ and$$
$$N_2(G) = \sum_{uv \in \varepsilon(con(G))} deg_G(u)\, deg_G(v).$$

For details on their history, mathematical properties and chemical applications see [1,2,3,4, 5,31] and the references cited therein. Now, we define these indices in a fuzzy graph.

Definition 2.9. *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph.*

- $M_1^F(G) := \sum\limits_{v_i \in V} deg_G^2(v_i);$
- $M_2^F(G) := \sum\limits_{v_i v_j \in \mathcal{E}} \mu(v_i v_j) deg_G(v_i) deg_G(v_j);$
- $F^F(G) := \sum\limits_{v_i \in V} deg_G^3(v_i);$
- $N_1^F(G) := \sum\limits_{u \in V(G)} deg_G(u)\, deg_{con(G)}(u);$
- $N_2^F(G) := \sum\limits_{uv \in \varepsilon(con(G))} \mu(uv) deg_G(u)\, deg_G(v).$

Definition 2.10. *Let* $A = [a_{ij}]_{m \times n}$. *Then, we define* $S(A) = \sum_{1 \le i \le m, 1 \le j \le n} a_{ij}$.

3. Main results

In this section, first, we define some fuzzy graph operations that were first studied in [16] by Mordeson and Peng in 1994. Later Sunitha and Vijayakumar [33] investigated the properties of compliments of fuzzy graphs with respect to these operations in 2002. Also, we define sum of the degrees of all neighbors of a vertex in a fuzzy graph and then we investigate some properties of it. We will establish relations between the fuzzy Zagreb indices and sum of the degrees of all neighbors of a vertex.

Definition 3.1. *Let* $G_1 = (V_1, \sigma_1, \mu_1)$ *and* $G_2 = (V_2, \sigma_2, \mu_2)$ *be two fuzzy graphs such that* $V_1 \cap V_2 = \varnothing$. *Union of two fuzzy graphs* G_1 *and* G_2 *is denoted by* $G_1 \cup G_2 = (V, \sigma, \mu)$ *such that* $V = V_1 \cup V_2$,

$$\sigma(v) = \begin{cases} \sigma_1(v) & , \quad v \in V_1 \\ \\ \sigma_2(v) & , \quad v \in V_2 \end{cases} \quad and \ \mu(uv) = \begin{cases} \mu_1(uv) & , \quad u, v \in V_1 \\ \mu_2(uv) & , \quad u, v \in V_2. \\ 0 & , \quad o.w \end{cases}$$

It is easy to see $G_1 \cup G_2$ *is a fuzzy graph.*

Definition 3.2. *Let* $G_1 = (V_1, \sigma_1, \mu_1)$ *and* $G_2 = (V_2, \sigma_2, \mu_2)$ *be two fuzzy graphs such that* $V_1 \cap V_2 = \phi$. *Sum of two fuzzy graphs* G_1 *and* G_2 *is denoted by* $G_1 \vee G_2 = (V, \sigma, \mu)$ *such that* $V = V_1 \cup V_2$,

$$\sigma(v) = \begin{cases} \sigma_1(v) & , \quad v \in V_1 \\ \\ \sigma_2(v) & , \quad v \in V_2 \end{cases}, \quad \mu(uv) = \begin{cases} \mu_1(uv) & , \quad u, v \in V_1 \\ \mu_2(uv) & , \quad u, v \in V_2 \\ k & , \quad u \in V_1, v \in V_2 \end{cases},$$

where $k = min\{\sigma_1(u), \sigma_2(v)\}$ *for every* $u \in V_1$ *and* $v \in V_2$.

Definition 3.3. *Let* $G_1 = (V_1, \sigma_1, \mu_1)$ *and* $G_2 = (V_2, \sigma_2, \mu_2)$ *be two fuzzy graphs. The Cartesian product of graphs* G_1 *and* G_2 *is denoted by* $G_1 \times G_2 = (V, \sigma, \mu)$ *is a fuzzy graph such that* $V = V_1 \times V_2$,

$\sigma((u, v)) = \sigma_1(u) \vee \sigma_2(v)$, *where* $\vee$ *is denoted maximum and*

$$\mu((u, v)(u', v')) = \begin{cases} \mu_2(vv') & , \quad if u = u' \\ \mu_1(uu') & , \quad if v = v'. \\ 0 & , \quad o.w \end{cases}$$

It is easy to show that $d_{G_1 \times G_2}((u, v)) = deg_{G_1}(u) + deg_{G_2}(v)$. *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph the neighbor of vertex* v *is denoted by* $N_G(v)$ *and is defined as follows.*

$$N_G(v) = \{u \in V \mid \mu(uv) > 0\}.$$

Let G *be a graph. Sum of the degrees of all neighbors of vertex* u *in* G *denoted by* $S_G(u)$ *and define as* $S_G(u) = \sum_{vu \in \mathcal{E}} deg_G v$. *Now, we will extend it to a fuzzy graph.*

Definition 3.4. *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph and* $u \in V$. *We define* $S_G^F(u) = \sum_{vu \in \mathcal{E}} \mu(vu) deg_G(v)$ *(*v *is a neighbor of vertex* u*).*

Theorem 3.5. *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph. Then*

$$\sum_{u \in V} S_G^F(u) = M_1^F(G).$$

Proof: *By Definition 3.4,*

$$\sum_{u \in V} S_G^F(u) = \sum_{u \in V} \sum_{uv \in \mathcal{E}} \mu(uv) deg_G(v) = \sum_{u \in V} \sum_{v \in V} \mu(uv) deg_G(v) = \sum_{v \in V} \sum_{u \in V} \mu(uv) deg_G(v) = \sum_{v \in V} deg_G^2(v) = M_1^F(G).$$

Theorem 3.6. *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph. Then*

$$\sum_{u \in V} deg_G(u) S_G^F(u) = 2M_2^F(G).$$

Proof: By definion of $S_G^F(u)$, we have:

$$\sum_{u \in V} deg_G(u) S_G^F(u) = \sum_{u \in V} deg_G(u) \sum_{uv \in \mathcal{E}} \mu(uv) deg_G(v) =$$

$$\sum_{u \in V} \sum_{uv \in \mathcal{E}} \mu(uv) deg_G(v) deg_G(u) =$$

$$2 \sum_{uv \in \mathcal{E}} \mu(uv) deg_G(u) deg_G(v) = 2M_2^F(G).$$

Theorem 3.7. *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph and* $con(G) = (V, \omega, \lambda)$ *the fuzzy* *congraph of* G. *If* G *has no cycles of size* 4, *then*

$$d_{con(G)}(v) = S_G^F(u) - \sum_{u \in V} \mu^2(uv), \quad v \in V.$$

Proof: $d_{con(G)}(v) = \sum_{u \in V} \lambda(uv) = \sum_{u \in V} min_{x \in H}\{\mu(vx).\mu(xu)\}$,
where $H = N_G(u) \cap N_G(v)$. Since G has no cycle of size *4*, it follows that $H \subseteq \{w\}$ for some $w \in V$. Therefore,

$$d_{con(G)}(v) = \sum_{vw,wu \in \mathcal{E}(G)} \mu(vw)\mu(wu) =$$
$$\sum_{vw \in \mathcal{E}(G)} \mu(vw) \sum_{u \in V} \mu(uw) - \sum_{w \in V} \mu^2(vw) =$$

$$S_G^F(w) - \sum_{w \in V} \mu^2(vw) = S_G^F(u) - \sum_{u \in V} \mu^2(vu).$$

Theorem 3.8. *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph and* A_F *be the adjacency of* G. *If* G *has no cycles of size* 4, *then*
$N_1^F(G) + S(A_F.(A_F \odot A_F)) = 2M_2^F(G)$.
Proof: By the definition of $N_1^F(G)$ and by the Theorem 3.7, we have

$$N_1^F(G) = \sum_{u \in V} deg_G(u) deg_{con(G)}(u) = \sum_{u \in V} deg_G(u)(S_G^F(u) - \sum_{v \in V} \mu^2(uv)) =$$

$$\sum_{u \in V} deg_G(u) S_G^F(u) - \sum_{u \in V} \sum_{v \in V} deg_G(u) \mu^2(uv)) =$$

$$2M_2^F(G) - \sum_{u \in V} \sum_{v \in V} \sum_{t \in V} \mu(tu)\mu^2(uv)) =$$
$$2M_2^F(G) - \sum_{u \in V} \sum_{t \in V} \sum_{v \in V} \mu(tv)\mu^2(uv)) =$$
$$2M_2^F(G) - S(A_F.(A_F \odot A_F)).$$

Hence, $N_1^F(G) + S(A_F.(A_F \odot A_F)) = 2M_2^F(G)$.

Since in the ordinary graph $A \odot A = A$, by the above theorem, and by [32, Lemma 2.3(2)], we deduce the following result.

Corollary 3.9. *Let* G *be a graph which has no cycles of size* 4. *Then,*

$$M_2(G) = \frac{1}{2}(N_1(G) + M_1(G)).$$

Theorem 3.10. *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph such that it has no cycles of size* 4, *then*

$$2S(con(G)) = \frac{1}{2} M_1^F(G) - S(A_F \odot A_F).$$

Proof: We have:

$$2S(con(G)) = \sum_{u \in V} deg_{con(G)}(v) = \sum_{u \in V} (S_G^F(u) - \sum_{v \in V} \mu^2(uv)) =$$
$$\sum_{u \in V} S_G^F(u) - \sum_{u \in V} \sum_{v \in V} \mu^2(uv))$$
$$M_1^F(G) - S(A_F \odot A_F).$$

From the above theorem, and by [32, Lemma 2.3(1)], we deduce the following result in a graph.

Corollary 3.11. *Let* G *be a* (p, q)*-graph and have no cycles of size* 4. *Also, let* $con(G)$ *be a* $(p, q^{*'})$*-graph. Then,* $q^{*'} = \frac{1}{2} M_1(G) - q$.

Lemma 3.12. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph such that G isn't a null fuzzy graph. Then*

$$\frac{\delta}{2} \leq \frac{M_2^F(G)}{M_1^F(G)} \leq \frac{\Delta}{2}.$$

Proof: Let $u \in V$ be arbitrary. It is clear that $\delta - 1 \leq deg_G u - 1 \leq \Delta - 1$. Since $S_G^F(u) > 0$, we have

$$(\delta - 1)S_G^F(u) \leq (deg_G(u) - 1)S_G^F(u) \leq (\Delta - 1)S_G^F(u).$$

So,

$$\sum_{u \in V} (\delta - 1)S_G^F(u) \leq \sum_{u \in V} (deg_G(u) - 1)S_G^F(u) \leq \sum_{u \in V} (\Delta - 1)S_G^F(u).$$

So,

$$\delta \sum_{u \in V} S_G^F(u) - \sum_{u \in V} S_G^F(u) \leq \sum_{u \in V} deg_G(u)S_G^F(u) - \sum_{u \in V} S_G^F(u)$$
$$\leq \Delta \sum_{u \in V} S_G^F(u) - \sum_{u \in V} S_G^F(u).$$

By Theorem 3.5 and 3.6,

$$\delta M_1^F(G) - M_1^F(G) \leq 2M_1^F(G) - M_1^F(G) \leq \Delta M_1^F(G) - M_1^F(G).$$

Hence,

$$\delta M_1^F(G) \leq 2M_2^F(G) \leq \Delta M_1^F(G),$$

and so

$$\frac{\delta}{2} \leq \frac{M_2^F(G)}{M_1^F(G)} \leq \frac{\Delta}{2}.$$

Theorem 3.13. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then*

$$\sum_{uv \in \mathcal{E}} \mu(uv)[S_G^F(u) + S_G^F(v)] = 2M_2^F(G).$$

Proof:

$$\sum_{uv \in \mathcal{E}} \mu(uv)[S_G^F(u) + S_G^F(v)] = \sum_{uv \in \mathcal{E}} \mu(uv)S_G^F(u) + \sum_{uv \in \mathcal{E}} \mu(uv)S_G^F(v) =$$
$$\frac{1}{2}\sum_{u \in V} \left(\sum_{v \in V} \mu(uv)\right)S_G^F(u) + \frac{1}{2}\sum_{v \in V} \left(\sum_{u \in V} \mu(uv)\right)S_G^F(v) =$$
$$\frac{1}{2}\sum_{u \in V} deg_G(u)S_G^F(u) + \frac{1}{2}\sum_{v \in V} deg_G(v)S_G^F(v) =$$

By Theorem 3.6,

$$\frac{1}{2} \times 2M_2^F(G) + \frac{1}{2} \times 2M_2^F(G) = 2M_2^F(G).$$

Let $S_G(u)$ be sum of the degrees of all neighbors of vertex u in graph G. The following result can be obtained from the above theorem.

Corollary 3.14. *Let G be a graph having no cycles of size 4. Then,*

$$\sum_{uv \in E} (S_G(u) + S_G(v)) = 2M_2(G).$$

Lemma 3.15. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then, for every $v \in V$ the following holds:*

$$\sum_{v_i v_j \in E} \mu(v_i v_j)(deg_G^k(v_i) + deg_G^k(v_j)) = \sum_{v_i \in V} deg_G^{k+1}(v_i).$$

In particular,

$$\sum_{v_i v_j \in E} \mu(v_i v_j)(deg_G(v_i) + deg_G(v_j)) = \sum_{v_i \in V} deg_G^2(v_i) = M_1^F(G),$$

and

$$\sum_{v_iv_j\in E}\mu(v_iv_j)(deg_G^2(v_i)+deg_G^2(v_j))=\sum_{v_i\in V}deg_G^3(v_i)=F^F(G).$$

Proof:

$$\sum_{v_iv_j\in E}\mu(v_iv_j)(deg_G^k(v_i)+deg_G^k(v_j))=\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\mu(v_iv_j)(deg_G^k(v_i)+deg_G^k(v_j))=$$

$$\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\mu(v_iv_j)deg_G^k(v_i)+\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\mu(v_iv_j)deg_G^k(v_j)=$$
$$\frac{1}{2}\sum_{i=1}^{p}deg_G^k(v_i)\sum_{j=1}^{p}\mu(v_iv_j)+\frac{1}{2}\sum_{j=1}^{p}deg_G^k(v_j)\sum_{i=1}^{p}\mu(v_iv_j)=$$
$$\frac{1}{2}\sum_{i=1}^{p}deg_G^k(v_i)deg_G(v_i)+\frac{1}{2}\sum_{j=1}^{p}deg_G^k(v_j)deg_G(v_j)=$$
$$\sum_{v_i\in V}deg_G^{k+1}(v_i).$$

Theorem 3.16. *Let $G=(V,\sigma,\mu)$ be a fuzzy graph such that it has no cycles of size 4, then*

$$\sum_{uv\in\mathcal{E}}(deg_G(u)S_G^F(v)+deg_G(v)S_G^F(u))=F^F(G)+2N_2^F(G).$$

Proof: By Definition 3.4, we have

$$\sum_{uv\in\mathcal{E}}(deg_G(u)S_G^F(v)+deg_G(v)S_G^F(u))=\sum_{uv\in\mathcal{E}}deg_G(u)\sum_{tv\in\mathcal{E}}deg_G(t)\mu(vt)+$$

$$\sum_{uv\in\mathcal{E}}deg_G(v)\sum_{su\in\mathcal{E}}deg_G(s)\mu(su)=$$
$$\sum_{uv\in\mathcal{E}}deg_G(u)[\sum_{vt\in\mathcal{E},t\neq u}deg_G(t)\mu(st)+deg_G(u)\mu(uv)]+$$
$$\sum_{uv\in\mathcal{E}}deg_G(v)[\sum_{su\in\mathcal{E},s\neq v}deg_G(s)\mu(su)+deg_G(v)\mu(uv)]=$$
$$\sum_{uv\in\mathcal{E}}\mu(uv)[deg_G^2(u)+deg_G^2(v)]+$$
$$\sum_{uv\in\mathcal{E}}\sum_{vt\in\mathcal{E},t\neq u}\mu(vt)deg_G(t)deg_G(u)+$$
$$\sum_{uv\in\mathcal{E}}\sum_{su\in\mathcal{E},s\neq v}\mu(us)deg_G(v)deg_G(s).$$

By the above lemma $\sum_{uv\in\mathcal{E}}\mu(uv)[deg_G^2(u)+deg_G^2(v)]=F^F(G)$. On the other hand, since G has no cycles of size 4, we have

$$\sum_{uv\in\mathcal{E}}\sum_{vt\in\mathcal{E},t\neq u}\mu(vt)deg_G(t)deg_G(u)+\sum_{uv\in\mathcal{E}}\sum_{su\in\mathcal{E},s\neq v}\mu(us)deg_G(v)deg_G(s)$$
$$=\sum_{ut\in con(G)}\mu(uv)\mu(vt)deg_G(t)deg_G(u)+\sum_{vs\in con(G)}\mu(uv)\mu(us)deg_G(v)deg_G(s)$$

By the Definition 2.9, the above relation is equal to the following relation

$$\sum_{ut\in con(G)}\lambda(us)deg_G(t)deg_G(u)+\sum_{vs\in con(G)}\lambda(us)deg_G(v)deg_G(s)=2N_2^F(G).$$

Therefore,

$$\sum_{uv\in\mathcal{E}}(deg_G(u)S_G^F(v)+deg_G(v)S_G^F(u))=F^F(G)+2N_2^F(G).$$

From the above theorem, we deduce the following result.

Corollary 3.17. *Let G be a graph having no cycles of size 4. Then,* $\sum_{uv\in E}(deg_G(u)S_G(v)+deg_G(v)S_G(u))=F+2N_2(G)$, *where* $F=\sum_{v\in V}deg(v)^3$.

Lemma 3.18. *Let $G_1=(V_1,\sigma_1,\mu_1)$ and $G_2=(V_2,\sigma_2,\mu_2)$ be two fuzzy graphs such that*

$V_1 \cap V_2 = \varnothing$, $|V_1| = p_1$, *and* $|V_1| = p_2$.

(a) $S^F_{G_1 \cup G_2}(u) = \begin{cases} S^F_{G_1}(u) & , \quad u \in V_1 \\ & ; \\ S^F_{G_2}(u) & , \quad u \in V_2 \end{cases}$

(b) $S^F_{G_1 \vee G_2}(u) = \begin{cases} S^F_{G_1}(u) + kp_2 deg_{G_1}(u) + 2kS(G_2) + k^2 p_1 p_2 & , \quad u \in V_1 \\ \\ S^F_{G_2}(u) + kp_1 deg_{G_2}(u) + 2kS(G_1) + k^2 p_1 p_2 & , \quad u \in V_2 \end{cases}$, *where* $k = min\{\sigma_1(u), \sigma_2(v)\}$ *for every* $u \in V_1$ *and* $v \in V_2$;

(c) $S^F_{G_1 \times G_2}(a,b) = S^F_{G_2}(b) + S^F_{G_1}(a) + 2deg_{G_1}(a) deg_{G_2}(b)$.

Proof: *(a)*

$$S^F_{G_1 \cup G_2}(u) = \sum_{vu \in V_1 \cup V_2} \mu_{G_1 \cup G_2}(uv) deg_{G_1 \cup G_2}(v) = \begin{cases} \sum_{uv \in \mathcal{E}(G_1)} \mu_{G_1}(uv) deg_{G_1}(v) & , \quad u \in V_1 \\ \\ \sum_{uv \in \mathcal{E}(G_2)} \mu_{G_2}(uv) deg_{G_2}(v) & , \quad u \in V_2 \end{cases} = \begin{cases} S^F_{G_1}(u) & , \quad u \in V_1 \\ \\ S^F_{G_2}(u) & , \quad u \in V_2 \end{cases}.$$

(b) Let $u \in V_1$. *Then,*

$$S^F_{G_1 \vee G_2}(u) = \sum_{uv \in \mathcal{E}(G_1 \vee G_2)} \mu_{G_1 \vee G_2}(uv) deg_{G_1 \vee G_2}(v) = \sum_{uv \in \mathcal{E}(G_1)} \mu_{G_1}(uv) deg_{G_1 \vee G_2}(v) + \sum_{u \in V_1, v \in V_2} k deg_{G_1 \vee G_2}(v) = \sum_{uv \in \mathcal{E}(G_1)} \mu_{G_1}(uv)[deg_{G_1}(v) + kp_2] + \sum_{v \in V_2} k[deg_{G_2}(v) + kp_1] = S^F_{G_1}(u) + kp_2 deg_{G_1}(u) + 2kS(G_2) + k^2 p_1 p_2.$$

Similarly, if $u \in V_2$, *then*

$$S^F_{G_1 \vee G_2}(u) = S^F_{G_2}(u) + kp_1 deg_{G_2}(u) + 2kS(G_1) + k^2 p_1 p_2.$$

(c)

$$S^F_{G_1 \times G_2}(a,b) = \sum_{(a,b)(c,d) \in \mathcal{E}(G_1 \times G_2)} \mu_{G_1 \times G_2}((a,b)(c,d)) deg_{G_1 \times G_2}(c,d) = \sum_{(a,b)(a,d) \in \mathcal{E}(G_1 \times G_2)} \mu_{G_2}(b,d)[deg_{G_1}(a) + deg_{G_2}(d)] + \sum_{(a,b)(c,b) \in \mathcal{E}(G_1 \times G_2)} \mu_{G_1}(a,c))[deg_{G_1}(c) + deg_{G_2}(b)] = \sum_{bd \in \mathcal{E}(G_2)} \mu_{G_2}(bd) deg_{G_2}(d) + \sum_{bd \in \mathcal{E}(G_2)} \mu_{G_2}(bd) deg_{G1}(a) + \sum_{ac \in \mathcal{E}(G_2)} \mu_{G_1}(ac) deg_{G_1}(c) + \sum_{ac \in \mathcal{E}(G_2)} \mu_{G_1}(ac) deg_{G2}(b) = S^F_{G_2}(b) + S^F_{G_1}(a) + 2deg_{G_1}(a) deg_{G_2}(b).$$

Corollary 3.19. *Suppose that* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *are* (p_1, q_1) *and* (p_2, q_2) *graphs, respectively, such that* $V_1 \cap V_2 = \varnothing$. *Then*

(1) $S_{G_1 \cup G_2}(u) = \begin{cases} S_{G_1}(u) & , \quad u \in V_1 \\ & ; \\ S_{G_2}(u) & , \quad u \in V_2 \end{cases}$

(2) $S_{G_1 \vee G_2}(u) = \begin{cases} S_{G_1}(u) + p_2 deg_{G_1}(u) + 2q_2 + p_1 p_2 & , \quad u \in V_1 \\ & ; \\ S_{G_2}(u) + p_1 deg_{G_2}(u) + 2q_1 + p_1 p_2 & , \quad u \in V_2 \end{cases}$

(3) $S_{G_1 \times G_2}(a,b) = S_{G_2}(b) + S_{G_1}(a) + 2deg_{G_1}(a) deg_{G_2}(b)$.

4. Conclusion and future work

Theoretical concepts of fuzzy graphs are highly utilized by computer science applications. Especiallyin research areas of computer science such as data mining, image segmentation, clustering, image capturingand networking. So, in this paper, we introduced some properties of $S_G^F(u)$ on fuzzy graphs and studied the relations between the Zagreb indices and the sum of the degrees of all neighbors of a vertex.

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