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Historical Development of some Fixed Point Results in Metric Space

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Abstract. In this paper, the historical account of fixed point results for single mapping in metric space has been provided. Though, there is a vast account of fixed point results for two or more mappings in the literature. It is mainly concentrated on single mapping due to our philosophical touch on Sthira Vindu (fixed point) and Kutastha Vindu in Vedanta philosophy.

Keywords: Fixed Point Theory, Kutastha Vindu, Vedanta philosophy, Banach contraction

AMS Mathematics Subject Classification (2010): 37C25

1. Introduction

Stefan Banach (1892-1945)

We will discuss the most basic fixed point theorem in analysis, known as the Banach Contraction Principle (BCP). It is due to Stefan Banach and appeared in his Ph.D. thesis 1920, (published in 1922). The Banach Contraction Principle was first stated and proved by Banach for the contraction maps in the setting of complete metric spaces. At the same time, the concept of abstract metric space was introduced by Hausdorff, which then provided the general framework for the principle for contraction mappings in a complete metric space. The Banach Contraction Principle can be applied to mappings which are differentiable, or more generally, Lipschitz continuous [22].

1.1. Basic definition and examples

Definition 1.1.1. [26] Let (X, d) be a metric space.The map $T : X \to X$ is said to be *Lipschitzian* if there exists a constant $k > 0$ (called Lipschitz constant) such that, $d(T(x), T(y)) \leq kd(x, y)$, for all $x, y \in X$.

A *Lipschitzian mapping* with a Lipschitz constant $k < 1$ is called *contraction.*

Definition 1.1.2. [22] Let (X, d) be a metric space with $I = [0,1]$. A mapping $W : X \times X \times I \to X$ is said to be a *convex structure* on X if for each $(x, y, \lambda) \in X \times X \times I$ and $z \in X$, We have, $d(z, W(x, y, \lambda)) \leq \lambda\, d(z, x) + (1 - \lambda) d(z, y)$.
Throughout let us write $W(x, y, \lambda) = \lambda\, x \oplus (1 - \lambda) y$ whenever the choice of the *convexity mapping* W is irrelevant. Moreover, if we have $d(\frac{1}{2}\, p \oplus \frac{1}{2}\, x,\ \frac{1}{2}\, p \oplus \frac{1}{2}\, y) \leq \frac{1}{2} d(x, y)$, for all $p, x, y \in X$, Then X is said to be a *hyperbolic metric* space.

Definition 1.1.3. Let (X, d) be a hyperbolic metric space. We say that X is *uniformly convex* (in short, UC) if for any $a \in X$, for every $r > 0$, and for each $\varepsilon > 0$

$$\delta\,(r, \varepsilon) = \inf\left\{1 - \frac{1}{r} d\left(\frac{1}{2} x \oplus \frac{1}{2} y, a\right);\ d(x, a) \leq r, d(y, a) \leq\ r, d(x, y) \geq r\varepsilon\right\} > 0.$$

These are the following properties,
(a) $\delta\,(r, 0) = 0$, and $\delta\,(r, \varepsilon)$. Is an increasing function of ε for every fixed r.
(b) For $r_1 \leq r_2$, there holds, $1 - \frac{r_2}{r_1}\left(1 - \delta\left(r_2, \varepsilon \frac{r_1}{r_2}\right)\right) \leq\ \delta\,(r_1, \varepsilon)$.
(c) If (X, d) is *uniformly convex*, then (X, d) is strictly convex, That is, whenever $d\left(\frac{1}{2} x \oplus \frac{1}{2} y, a\right) = d(x, a) = d(y, a)$, For any $x, y, a \in X$, then we must have $x = y$.

The concept of *p-uniform convexity* was used extensively by Xu [239], its nonlinear version for $p = 2$ is as follows:

Definition 1.1.4. [10] The pair (X, d) is said to be *2-uniformly convex* if $c_X = \inf\left\{\frac{\Psi\,(r, \varepsilon)}{r^2 \varepsilon^2}; r > 0, \varepsilon > 0\right\} > 0$. It's noted that (X, d) is *2-uniformly convex* if and only if
$inf\left\{\frac{\delta\,(r, \varepsilon)}{\varepsilon^2}; r > 0, \varepsilon > 0\right\} > 0$.

Definition 1.1.5. A mapping $T : C \to C$ (a subset of X) is said to be *uniformly Lipschitzian* if there exists a nonnegative number k such that $d(T^n(x), T^n(y)) \leq kd\,(x, y)$, For all $x, y \in C$, and $n \geq 1$.
The smallest such constant k will be denoted by $\lambda\,(T)$.

Definition 1.1.6. Let X be an abstract set. A family Σ of subsets of X is called a *convexity structure* if
(a) The empty set $\emptyset \in \Sigma$;
(b) $X \in \Sigma$;
(c) Σ is closed under arbitrary intersections.
The notion of *hyper convexity* is due to Aronszajn and Panitchpakdi;

Definition 1.1.7. [13] The metric space M is said to be *hyperconvex* if for any collection of points $\{x_\alpha\}_{\alpha \in \Gamma}$ in M and positive numbers $\{r_\alpha\}_{\alpha \in \Gamma}$ such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ for any α *and* β in Γ, we must have $\bigcap_{\alpha\in\Gamma} B(x_\alpha, r_\alpha) \neq \emptyset$. Clearly the real line R is *hyperconvex*.

In 1996, Khamsi introduced a new concept called *1-local retract*, by using the non-expansive extension as follows;

Definition 1.1.8. [22] Let X be a metric space. A Subset $N \subset X$ is called a *1-local retract* of X if for any point $x \in X\backslash N$, there exists a *non-expansive retraction* $R : X \cup \{x\} \to N$.

Definition 1.1.9. [22] Let (X, d) be a metric space and, let $T : X \to X$ be a mapping.
(a) A point $x \in X$ is called a fixed point of T if $x = T(x)$.
(b) T is called contraction if there exists a fixed constant $h < 1$
Such that $d\big(T(x), T(y)\big) \leq h\, d(x, y)$, for all $x, y \in X. \dots$ (2.1)
A contraction mapping is also known as Banach contraction. If we replace the inequality (2.1) with strict inequality and $h = 1$, then T is called contractive (or strict contractive). If (2.1) holds for $h = 1$, then T is called nonexpansive; and if (2.1) holds for fixed $h < \infty$, Then T is called *Lipschitz continuous*. Clearly, for the mapping T, the following obvious implications hold:
Contraction $\Rightarrow$ *contractive* $\Rightarrow$ *nonexpansive* $\Rightarrow$ *Lipschitz continuous*

Example 1.1.1. (a) Let $T : [0,2] \to [0,2]$ be defined by

$$T(x) = \begin{cases} 0, & x \in [0,1], \\ 1, & x \in [1,2]. \end{cases}$$

Then, $T^2(x) = 0$ for all $x \in [0,2]$, and so, $T^2$is a contraction on [0,2] .
Its noted T is not continuous and thus not a contraction map.

Definition 1.1.10. [10] (a) A real-valued function φ defined on X is said to be *lower semi continuous* at x if for any sequence $\{x_n\} \subset X$, we have $x_n \to x \in X \Rightarrow \varphi(x) \leq \lim_{n\to\infty} inf \varphi\,(x_n)$
(b)A single-valued self-mapping T on a metric space (X, d) is said to be Caristi mapping if there exists a lower semi continuous function $\phi : X \to R^+$ Such that $d\big(x, T(x)\big) \leq \varphi(x) - \varphi\big(T(x)\big)$, For all$x \in X$.

Example 1.1.2. Each Banach contraction mapping T on a metric space (X, d) is a Caristi mapping with a function $\varphi\,(x) = \frac{1}{1-h} d\big(x, T(x)\big)$, for all $x \in X$, Where h is a contraction constant. Clearly, φ is a continuous real valued function on X and $\varphi\,(Tx) \leq \frac{h}{1-h} d(x, Tx) = h\,\varphi\,(x)$. Its noted that for all $x \in X$,
$d\big(x, T(x)\big) = (1-h)\phi\,(x) = \phi\,(x) - h\,\phi\,(x) \leq \phi\,(x) - \phi\,(Tx)$,
That is, T is a Caristi mapping.

Definition 1.1.11. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be contractive if there exists $r \in [0,1)$ such that $d(Tx, Ty) \leq r\, d(x, y)$ for all $x, y \in X$. Such a mapping is also called *r-contractive*.
In 2010, Iemoto, Takahashi and Ying also introduced the following class of mappings of X into itself. Let p be a *w-distance* on X.

Definition 1.1.12. [23] A mapping $T : X \to X$ is called *p-contractively* non spreading if there exists $\alpha \in [0,1/2)$ such that $p(Tx,Ty) \leq \alpha\{p(Tx,y) + p(x,Ty)\} \forall x,y \in X$.

Definition 1.1.13. [13] A mapping $T : X \to X$ is said to be *Kannan* if there exists $\alpha \in 0, \frac{1}{2})$ such that $d(Tx,Ty) \leq \alpha\{d(x,Tx) + d(y,Ty)\}$ for all $x,y \in X$.

Definition 1.1.14. ([24], [23]) A mapping $T : X \to X$ is said to be *contractively non spreading* if there exists $\beta \in [0, \frac{1}{2})$ such that $d(Tx,Ty) \leq \beta\{d(x,Ty) + d(y,Tx)\}$ for all $x,y \in X$.

Definition 1.1.15. [10] A mapping $T : X \to X$ is called *contractively hybrid* if there exists $\gamma \in [0, \frac{1}{3})$ such that $d(Tx,Ty) \leq r\{d(Tx,y) + d(Ty,x) + d(x,y)\}$Forall $x,y \in X$.
In 1998, Shioji, Suzuki and Takahashi also introduced the sets $WC_2(X), WC_0(X), WK_1(X), WK_2(X)$ and $WK_0(X)$ of mappings of X into itself as follows:

Definition 1.1.16. [20] $T \in WC_2(X)$ if and only if there exist $p \in W(X)$ and $r \in [0,1)$ such that $p(Tx,Ty) \leq rp(y,x)$ for all $x,y \in X$; $T \in WC_0(X)$ if and only if there exist $p \in W_0(X)$ and $r \in [0,1)$ such that $p(Tx,Ty) \leq rp(x,y)$ for all $x,y \in X$; $T \in WK_1(X)$ if and only if there exist $p \in W(X)$ and $\alpha \in [0,1/2)$ such that $p(Tx,Ty) \leq \alpha\{p(Tx,x) + p(Ty,y)\}$ for all $x,y \in X$; $T \in WK_2(X)$ if and only if there exist $p \in W(X)$ and $\alpha \in [0,1/2)$ such that $p(Tx,Ty) \leq \alpha\{p(Tx,x) + p(y,Ty)\}$ for all $x,y \in X$; $T \in WK_0(X)$ if and only if there exist $p \in W_0(X)$ and $\alpha \in [0,1/2)$
Such that $p(Tx,Ty) \leq \alpha\{p(Tx,x) + p(Ty,y)\}$ for all $x,y \in X$. In particular, a mapping $T \in WK_1(X)$ is called *p-Kannan*.

Definition 1.1.17. A function $p: X \times X \to R^+$is called a *w-distance* on X if it satisfies the following for any $x,y,z \in X$:
(w_1) $p(x,z) \leq p(x,y) + p(y,z)$;
(w_2) a map $p(x,.) : X \to [0,\infty)$ is *lower semi continuous*;
(w_3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that
$p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$.
In 2001, Suzuki in generalizing the concept of *w-distance*, introduced the following notion of *τ-distance* on metric spaces.

Definition 1.1.18. [26] A function $p : X \times X \to R^+$ is said to be a *τ-distance* on X if it satisfies the following conditions for any $x,y,z \in X$, Such
(τ_1) $p(x,z) \leq p(x,y) + p(y,z)$; (τ_2) $\eta(x,0) = 0$, and $\eta(x,t) \geq t$ for all $x \in X$ and $t \geq 0$,
and η is *concave* and *continuous* in its second variable;
(τ_3) ${}^{Lim}_{n}\ x_n = x$ and ${}^{Lim}_{n}\sup\{\eta(z_n, P(z_n,x_n)): m \geq n\} = 0$
imply $P(u,x) \leq {}^{Lim}_{n} inf\ P(u,x_n)$ for all $u \in X$;
(τ_4) ${}^{Lim}_{n}$ $\sup\{P(x_n,y_n):\ m \geq n\} = 0$ and ${}^{Lim}_{n}$ $\eta(x_n,t_n) = 0$
implies ${}^{Lim}_{n}$ $\eta(y_n,t_n) = 0$;
(τ_5) ${}^{Lim}_{n}$ $\eta(z_n,\ P(z_n,\ x_n) = 0$ and ${}^{Lim}_{n}$ $\eta(z_n, P(z_n,\ y_n) = 0$

implies $\lim_{n} d(x_n, y_n) = 0$

Definition 1.1.19. Let (X, d) be a metric space and let $T : X \to 2^X$.
(a) An element $x \in X$ is called a fixed point of a *multivalued mapping* T if $x \in T(x)$. We denote $Fix(T) = \{x \in X : x \in T(x)\}$.
(b) A sequence $\{x_n\}$in X is said to be an *iterative* sequence of T at $x_0 \in X$
If $x_n \in T(x_{n-1})$ for all $n \in N$.
(c) T is said to be a *contraction* if for a fixed constant $h < 1$ and for each $x, y \in X$,

$$H(T(x), T(y)) \leq hd\,(x, y).$$

Such a mapping T is also known as *Nadler contraction*.

Definition 1.1.20. Let X be a metric space. A function $T : X \to R$ is said to be *lower semi continuous* at a point $x \in X$ if $T(x) \leq \lim_{n\to\infty} inf\ T(x_n)$ whenever$x_n \to x$ as $n \to \infty$. T is said to be *lower semi continuous* on X if it is *lower semi continuous* at each point of X. A function $T : X \to R$ is said to be *upper semi continuous* at a point $x \in X$ If $T(x) \geq \lim_{n\to\infty} sup\ T(x_n)$ whenever $x_n \to x$ as $n \to \infty$.
T is said to be *upper semi continuous* on X if it is *upper semi continuous* at each point of X.

Definition 1.1.21. For a given $\varepsilon > 0$, an element x_ε is said to be an approximate ε -*solution* of the following minimization problem $\inf_{x\in X} T(x)$, if $\inf_X T \leq (x_\varepsilon) \leq \inf_X T + \varepsilon$, Where $\inf_X T = \inf_{x\in X} T(x)$.

Definition 1.1.22. Let X be a metric space. A function $T : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *lower semi continuous* from above at a point $x \in X$ if $x_n \to x\ as\ n \to \infty$ and $T(x_1) \geq T(x_2) \geq \cdots T(x_n) \geq \cdots$
imply that $T(x) \leq \lim_{x\to\infty} T(x_n)$.

Definition 1.1.23. Let (X, d) be a metric space. For any $x, y \in X$, the *segment* between x and y is defined by $[x, y] = \{z \in X : d(x, z) + d(z, y) = d(x, y)\}$.

Definition 1.1.24. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a *directional contraction* if
(i) T is continuous, and
(ii) There exists $\alpha \in (0,1)$ such that for any $x \in X$ with $T(x) \neq x$, There exists $z \in [x, T(x)]\backslash\{x\}$ such that $d(T(x), T(z)) \leq \alpha\, d(x, z)$.

Definition 1.1.25. Let K be a nonempty set, $T : K \times K \to R$ be a bifunction and $\varepsilon > 0$ be given. The element $\bar{x} \in K$ is said to be an ε -*solution* of Ekeland's Principal if $T(\bar{x}, y) \geq -\varepsilon\, d(\bar{x}, y)\ for\ all\ y \in K$. It is called strictly ε -solution of Ekeland's Principal if the above inequality is strictfor all $x \neq y$.

1.2. Some fixed point results in complete metric space

In 1922, the following theorems of Banach Contraction Principle were first stated and proved by Banach for the contraction maps in the setting of complete metric spaces.

Theorem 1.2.1. ([4], [22]) (*Banach Contraction Principle*). Let (X, d) be a complete metric space, then each contraction map $T : X \to X$ has a unique fixed point.
Proof: Let h be a contraction constant of the mapping T. We will explicitly construct a sequence converging to the fixed point. Let x_0 be an arbitrary but fixed element in X.
Define a sequence of iterates $\{x_n\}$in X by $x_n = T(x_{n-1})$ $(= T^n(x_0))$, for all $n \geq 1$. Since T is a contraction, we have $d(x_n, x_{n+1}) = d\big(T(x_{n-1}), T(x_n)\big) \leq hd(x_{n-1}, x_n)$, for any $n \geq 1$.
Thus, we obtain $d(x_n, x_{n+1}) \leq h^n d(x_0, x_1)$, for all $n \geq 1$.
Hence, for any $m > n$, we have $d(x_n, x_m) \leq (h^n + h^{n+1} + \cdots + h^{m-1}) d(x_0, x_1) \leq \frac{h^n}{1-h} d(x_0, x_1)$.
We deduce that $\{x_n\}$ is Cauchy sequence in a complete space X.
Let $x_n \to p \in X$. Now using the continuity of the map T, we get $p = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1}) = T(p)$. Finally, to show T has at most one fixed point in X,
Let p and q be fixed points of T. Then, $d(p, q) = d(T(p), T(q)) \leq hd(p, q)$. Since $h < 1$, we must have $p = q$. This completes the proof.

Theorem 1.2.2. Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction mapping, with Lipschitz constant $k < 1$. Then, T has a unique fixed point ω in X, and for each $x \in X$,
We have $\lim_{x\to\infty} T^n(x) = \omega$. Moreover for each $x \in X$, We have, $d(T^n(x), \omega) \leq \frac{k^n}{1-k} d(T(x), x)$.
An easy implication of the Banach Contraction Principle are the following theorem [22].

Theorem 1.2.3. Suppose (X, d) is a complete metric space and suppose $T : X \to X$ is a mapping for which T^N is a contraction mapping for some positive integer $N \geq 1$. Then T has a unique fixed point.

The following theorem is related to complete metric space.

Theorem 1.2.4. Let (X, d) be a compact metric space with $T: X \to X$ satisfying
$d\big(T(x), T(y)\big) < d(x, y)$For $x, y \in X$ and $x \neq y$. Then T has a unique fixed point in X.

Theorem 1.2.5. Let (X, d) be a complete metric space and
let $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$, where $x_0 \in X$ and $x > 0$.
It may be the case that $T : X \to X$ is not a contraction on the whole space X, but rather a contraction on some neighborhood of a given point. In this case the result as follows [22]:

Theorem 1.2.6. Let (X, d) be a complete metric space and Let $B_r(y) = \{x \in X : d(x, y) < r\}$, where $y \in X$ and $r > 0$. Let $f : B_r(y) \to X$ be a contraction map with contraction constant $h < 1$.further, assume that $d(y, T(y)) < r(1 - h)$. Then, T has a unique fixed point in $B_r(y)$.

In 1930, Caccioppoli extended the Banach Contraction Principal as follows;

Theorem 1.2.7. [22] Let (X,d) be a complete metric space and let $T: X \to X$ be a mapping such that for each $n \geq 1$, there exists a constant $c_n$such that $d\big(T^n(x),T^n(y)\big) \leq c_n\, d(x,y), for\ all\ x,y \in X,$
where $\sum_{n=1}^{\infty} c_n < \infty$. Then, T has a unique fixed point.

In 1968, Bryant extended Banach Contraction Principle as follows ;

Theorem 1.2.8. [26] Let (X,d) be a complete metric space and let $T: X \to X$ be a mapping such that for some positive integer n, T^n is contraction on X, then T has a unique fixed point.

Theorem 1.2.9. Let (X,d) be a complete metric space. A map $T: X \to X$ (not necessarily continuous).
Suppose the following condition holds:

$$\Big\{foreach\ \epsilon > 0\ thereis\ a\ \delta(\epsilon) > 0\ \ suchthat\ if\ d\big(x,T(x)\big) < \delta(\epsilon),\ \ thenT\big(B(x,\epsilon)\big) \subseteq B(x,\epsilon);\ hereB(x,\epsilon) = \{y \in X : d(x,y) < \epsilon\}\Big\}$$

If for some $u \in X$ we have $\operatorname{Lim}_{n\to\infty} d\,\big(T^n(u),T^{n+1}(u)\big) = 0$, then the sequence $\{T^n(u)\}$converges to a fixed point of T.

In 1962E.Rakotch first generalization of Banach Contraction Principle as follows;

Theorem 1.2.10. [10] Let (X,d) be a complete metric space, and suppose that $T: X \to X$, satisfies $d(\,T\,(x),T\,(y)) \leq \eta(d(x,y))d(x,y)$, for all$x,y \in X$, Where η is a decreasing function on $R+\ to$ $[0,1)$.
Then, T has a unique fixed point.

In 1969, Boyd D.W. and Wong J. S. W. more generalize theorem as follows;

Theorem 1.2.11. [4] Let (X,d) be a complete metric space, And suppose that $T: X \to X$ satisfie $d(\,T\,(x),T\,(y)) \leq \psi(d(x,y))$, For all $x,y \in X$, Where $\psi: R \to [0,\infty)$ is *upper semi continuous* from the right, That is, for any sequence $t_n \downarrow t \geq 0 \Rightarrow \lim_{n\to\infty} sup\psi\,(t_n) \leq \psi(t)$. And satisfies $0 \leq \psi(t) < t$ for $t > 0$, then, T has a unique fixed point.

In 1969 Meir A. and Keeler E. extended Boyd and Wong theorem is as follows;

Theorem 1.2.12. [22] Let (X,d) be a complete metric space, and suppose that $T: X \to X$ satisfies the condition: For each $\varepsilon > 0$, there exists $\delta > 0$. Such that for all$x,y \in X$, $\varepsilon \leq d(x,y) \leq \varepsilon + \delta$
$\Rightarrow d\big(T(x),T(y)\big) \leq \varepsilon$. Then, T has a unique fixed point.

In 1974, Ciric has generalized Banach Contraction Principle as follows;

Theorem 1.2.13. Let (X,d)be a complete metric space, and let $T: X \to X$ be a quasi-contraction, that is, for a fixed constant

$h < 1, d(T(x), T(y)) \leq h\, max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$,
For all $x, y \in X$. Then, T has a unique fixed point.

In 1975, Matkowski extended by replacing ψ Theorem 2.2.11 as follows;

Theorem 1.2.14. Let (X, d) be a complete metric space, and suppose that $T : X \to X$ Satisfies $d(T(x), T(y)) \leq \psi(d(x, y))$, for all $x, y \in X$, where $\psi: (0, \infty) \to (0, \infty)$ is monotone non decreasing and satisfies $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. Then, T has a unique fixed point.

In 1976, Caristi proved a unique Fixed Point Results in complete metric space related to Banach Contraction Principle as follows;

Theorem 1.2.15. [10] Let (X, d) be a complete metric space. Then, each Caristi map $T : X \to X$ has a fixed point.

In 2001, Rhoades extended and improved in metric space of the generalization of Alber [22] in Hilbert space as follows;

Theorem 1.2.16. [3] Let (X, d)be a complete metric space, and suppose that $T : X \to X$ satisfies the following inequality $d(T(x), T(y)) \leq d(x, y) - \psi(d(x, y))$, for all $x, y \in X$, where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$. Then, T has unique fixed point.

In 2003, Kirk W.A. obtained the asymptotic version of Boyd and Wong [4] as follows;

Theorem 1.2.17. [13] Let (X, d) be a complete metric space, and suppose that $T : X \to X$ Satisfies $d(T^n(x), T^n(y)) \leq \psi_n(d(x, y))$, for all $x, y \in X$, where $\psi_n: [0, \infty) \to [0, \infty)$ are continuous and $\psi_n \to \psi \in \Psi$ uniformly. Further, assume that some orbit of T is bounded. Then, T has a unique fixed point.

In 2008, Dutta P. N. and Chaudhary B. S. generalized Theorem 2.2.15 is as follows;

Theorem 1.2.18. [20] Let (X, d) be a complete metric space, and suppose that $T : X \to X$ satisfies the following inequality, $\varphi\left(d(T(x), T(y))\right) \leq \varphi\left(d(x, y)\right) - \psi\left(d(x, y)\right)$, For all $x, y \in X$, where both the functions $\varphi, \psi : [0, \infty) \to [0, \infty)$ are continuous and nondecreasing such that, $\psi(t) = 0 = \varphi(t)$ If and only if $t = 0$. Then, T has unique fixed point.

In 2011, Choudhury, Konarb, Rhoades and Metiya established more general result is as follows:

Theorem 1.2.19. [4] Let (X, d)be a complete metric space, and suppose that $T : X \to X$ satisfies the following inequality

$$\phi(d(T(x), T(y))) \leq \phi(m(x, y)) - \psi(max\{d(x, y), d(y, T(y))\}),$$

where $m(x,y) = max\left\{d(x,y), d\big(x,T(x)\big), d\big(y,T(y)\big), \frac{1}{2}\left[d\big(x,T(y)\big) + d(y,T(x))\right]\right\}$
For all $x, y \in X$, and $\phi, \psi : [0,\infty) \to [0,\infty)$ Are functions such that ϕ is alternating distance and ψ is continuous with $\psi(t) = 0$ If and only if $t = 0$. Then, T has unique fixed point.

A direct consequence of the Theorem 1.2.19 is the as follows;

Corollary 1.2.1. [22] Let (X,d) be a complete metric space, and suppose that $T : X \to X$ satisfies the following inequality for all $x, y \in X$,
$\phi\left(d\big(T^n(x), T^n(y)\big)\right)\phi\ \left((max\left\{d(x,y), d\big(x,T^n(x)\big), d\big(y,T^n(y)\big), \frac{1}{2}\left[d\big(x,T^n(y)\big) + d(y,T^n(x)\}\right] - \psi(\max\{d(x,y), d(y,T^n(y)\right\}\right)$
where n is a positive integer. And $\phi, \psi : [0,\infty) \to [0,\infty)$. Are functions such that ϕ alternating distance and ψ is continuous with $\psi(t) = 0$ If and only if $t = 0$. Then, T has unique fixed point.
In 2008, T. Suzuki gave a new type of generalization of the Banach Contraction Principle as follows,

Theorem 1.2.20. [20] Let(X,d) be a complete metric space, and suppose that$T : X \to X$. Define a non-increasing function $\psi : [0,1) \to (1/2,1]$ by

$$\psi(h) = \begin{cases} 1 & if \quad 0 \le h \le \dfrac{(\sqrt{5}-1)}{2}, \\ \dfrac{1-h}{h^2} & if \dfrac{(\sqrt{5}-1)}{2} < h < \dfrac{1}{\sqrt{2}}, \\ \dfrac{1}{1+h} & if \dfrac{1}{\sqrt{2}} \le h < 1. \end{cases}$$

Assume that there exists$h \in [0,1)$, Such that
$\psi(h)d\big(x,T(x)\big) \le d(x,y) \ \Rightarrow d\big(T(x),T(y)\big) \le d(x,y)$, For all$x, y \in X$.
Then, T has a unique fixed point.

2. Conclusion

The fixed point theory in metric space with different contraction condition and its generalized form are important as extension of famous Banach contraction principals and for its applications to other disciplines.

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REFERENCES

1. A.Meir and E.Keeler, A theorem on contraction mappings, *J. Math. Anal. Appl.*, 28 (1969) 326–329.

2. A.P.Panta, et al., A review of vacation queuing models in different framework, *Annals of Pure and Applied Mathematics*, 24(2) (2021) 99-121.
3. B.E.Rhoades, Two general fixed point theorems, *Kobe University mathematics seminar notes*, 5 (1977) 199-200.
4. B.S.Choudhury, P.Konarb, B.E.Rhoades and N.Metiya, Fixed point theorems for generalized weakly contractive mappings, *Nonlinear Anal.*, 74 (2011) 2116–2126.
5. D.F.Bailey, Some theorems on contractive mappings, *J. Indian Math. Soc.*, 41 (1966) 101-106.
6. D.Panthi, A Meir-Keeler type common fixed point result in dislocated metric space, *Journal of Nepal Mathematical Society*, 1(2) (2018) 53-59.
7. D.Panthi, Common fixed point theorems for compatible mappings in dislocated metric space, *International Journal of Mathematical Analysis*, 9 (45) (2015) 2235-2242.
8. D.Panthi and P.K.Jha, A short survey on fixed point results and applications, *International Journal of Statistics and Applied Mathematics*, 2(5) (2017) 25- 34.
9. D.W.Boyd, and J.S.Wong, Nonlinear contractions, *Proc. Am. Math. Soc.*. 20 (1969) 458–464.
10. E.Rakotch, A note on contractive mappings, *Proc. Am. Math. Soc.*, 13 (1962) 459–465.
11. F.E.Browder, Non expansive nonlinear operators in Banach Space, *Proceedings of the National Academy of Science of the United States of America*, 54 (1965) 1041-1044.
12. J.Caristi, Fixed point theorem for mapping satisfying inwardness conditions, *Trans. Am. Math. Soc.*, 215 (1976) 241–251.
13. J.Matkowski, Nonlinear contractions in metrically convex spaces, *Publ. Math. Debrecen*, 45 (1994) 103–114.
14. K.Atkinson, and W.Han, *Theoretical Numerical Analysis: A Functional Analysis Framework*, Springer, USA, 2009.
15. K.Goebel and S.Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Series of Monographs and Text books in Pure and Applied Mathematics, vol. 83. Marcel Dekker, New York (1984).
16. K.Hasegawa, T.Komiya and W.Takahashi, Fixed point theorems for general contractive mappings in metric spaces and estimating expressions, *Sci. Math. Jpn.*, 74 (2011) 15–27.
17. K.Jha, et al., A common fixed point theorem for reciprocal continuous compatible mappings in metric space, *Annals of Pure and Applied Mathematics*, 5(2) (2014) 120-124.
18. M.A.Khamsi, One-local retract and common fixed point for commuting mappings in metric spaces", *Nonlinear Anal.*, 27 (1996) 1307–1313.
19. M.S.Khan, A fixed Point theorem for metric spaces, *Riv. Mat. Univ. Parma*, 4 (1977) 53 -57.
20. N.Shioji, T. Suzuki and W.Takahashi, Contractive mappings, Kannan mappings and metric completeness, *Proc. Amer. Math. Soc.*, 126 (1998) 3117–3124.
21. P.N.Dutta and B.S.Choudhury, A generalisation of contraction principle in metric spaces, *Fixed Point Theory, Appl.*, 2008, Article ID 406368 (2008).
22. S.Almezel, Q.A.Ansari and M.A.Khamsi, *Topics in Fixed Point Theory*, Springer, Switzerland (2014).

23. S.Iemoto, W.Takahashi and H.Yingtaweesittikul, *Nonlinear operators, fixed points and completeness of metric spaces, in Fixed Point Theory and its Applications,* Yokohama Publishers, Yokohama, (2010) 93–101.
24. S.K.Chaterjee, Fixed point theorems, *Compts. Rend. Acad. Bulgeria Sci.,* 25 (1972) 727-730.
25. T.Suzuki, Generalized distance and existence theorems in complete metric spaces, *J. Math. Anal. Appl.,* 253 (2001) 440–458.
26. V.W.Bryant, A remark on a fixed point theorem for iterated mappings, *Am. Math. Mon.,* 75 (1968) 399–400.