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On the Diophantine Equation $4^x + n^y = z^2$

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Abstract. Let n be a positive integer where $n \equiv 1(\bmod 15)$. In this paper we shown that the Diophantine equation $4^x + n^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.

Keywords: Diophantine equation, Quadratic residue, non-negative integer

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1. Introduction

In the past, there was a lot of interest in studying the solution of Diophantine equations. The general form of the Diophantine equation is $a^x + b^y = c^z$ which has been studied in [4]. In 2008, Pumnea and Nicoar [8] studied Diophantine equations of the form $a^x + b^y = z^2$, for example: $2^x + 7^y = z^2, 2^x + 11^y = z^2$ and $2^x + 13^y = z^2$. Many authors also studied some particular cases of the Diophantine equation $4^x + b^y = z^2$, where b is a fixed number and b is a prime number. In 2011, Suvarnaman, Singta and Chotchaisthit [12] showed that Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no solution in non-negative integers. The following year, Chotchaisthit [3] showed that the Diophantine equation $4^x + p^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers and p is a positive prime. In 2014, Sroysang [11] established that the Diophantine equation $4^x + 10^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers. In 2016, Srisarakham and Thongmoon [10] solved that the Diophantine equation $48^x + 84^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1, 0, 7)$. In 2018, Kumar, Gupta and Kishan [5] showed that the Diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$ have no solution where x, y and z are non-negative integers. In the same year, Lu [6] investigated the equation of the form $q^x + p^y = z^2$ with q and p are primes. Particularly, Lu considered the equations $3^x + p^y = z^2$ where $p \equiv 5(\bmod 12)$ and $3^x + b^y = z^2$ where $b \equiv 1(\bmod 4)$ and

$p \equiv 5 (\bmod 12)$ or $p \equiv 7 (\bmod 12)$. In the next year, Burshtein [1] established some non-negative solutions for the Diophantine equation $3^x + p^y = z^2$ where p is an odd prime number and $x + y \leq 8$. Later, [2] proved that the equation $8^x + 9^y = z^2$ has no solution when x, y and z are positive integers by utilizing the last digits of the powers $8^x, 9^y$. In 2021, Moonchaisook [7] considered the non-linear Diophantine equation $p^x + \left(p + 4^n\right)^y = z^2$ has no solution where $p > 3, p + 4^n$ are primes.

In this paper, we consider the Diophantine equation $4^x + n^y = z^2$, where $n \equiv 1 (\bmod 15)$ and x, y, z are non-negative integers. Here we will study all the possible causes and we will use a quadratic residue of n.

2. Preliminaries

Let p be an odd prime and a be a positive integer where $gcd\left(a, p\right) = 1$. If the quadratic congruence $x^2 \equiv a \left(\bmod p\right)$ has a solution, then a is said to be *a quadratic residue of* p. Otherwise, a is called *a quadratic non-residue of* p. In 1798 Adrien-Marie Legendre [9] introduced the Legendre symbol $\left(\frac{a}{p}\right)$ which is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & ; \textit{if } a \textit{ is a quadratic residue of } p. \\ -1 & ; \textit{if } a \textit{ is a quadratic non-residue of } p. \end{cases}$$

In this paper, using the following symbols;

Lemma 2.1. The Diophantine equation $4^x + 1 = z^2$ has no non-negative integer solution where x and z are non-negative integers.

Proof: Let x and z are non-negative integers. If $x = 0$, then $z^2 = 2$, which is impossible. If $x = 1$, then $z^2 = 5$, which is impossible. If $x > 1$, then $4^x + 1 = z^2$. Since $4^x \equiv 1 (\bmod 3)$, thus $z^2 = 4^x + 1 \equiv 2 (\bmod 3)$ but $\left(\frac{2}{3}\right) = -1$, this equation has no solution.

Let $n \equiv 1 (\bmod 15)$. We get $15 | n - 1$ or $n - 1 = 15k$ for some integers k. Get $n = 15k + 1 = 3(5k) + 1 = 5(3k) + 1$ so $n \equiv 1 (\bmod 3)$ and $n \equiv 1 (\bmod 5)$. In this paper, we assume that n is a non-negative integer.

Lemma 2.2. Let n be a positive integer with $n \equiv 1 (\bmod 15)$. The Diophantine equation $1 + n^y = z^2$ has no non-negative integer solution y and z.

Proof: Let y and z are non-negative integers and n be a positive integer with $n \equiv 1 (\bmod 15)$ is clear that $n \equiv 1 (\bmod 3)$ and $n \equiv 1 (\bmod 5)$. We divide it into two cases as follows:

Case 1: if $y = 0$, then $2 = z^2$ is impossible.

Case 2: if $y \geq 1$, then $1 + n^y = z^2$. Since $n \equiv 1 \pmod 5$, thus $n^y \equiv 1 \pmod 5$ and $z^2 = 1 + n^y \equiv 2 \pmod 5$ but $\left(\frac{2}{5}\right) = -1$.

3. Main theorem

Theorem 3.1. Let n be a positive integer where $n \equiv 1 \pmod{15}$. The Diophantine equation $4^x + n^y = z^2$ has no non-negative integer solution x, y and z.

Proof: Let n be a positive integer where $n \equiv 1 \pmod{15}$, and x, y, z are non-negative integers. We divide it into three cases as follows:

Case 1: $x = 0$, by Lemma 2.2, there is no non-negative integer solution.

Case 2: $y = 0$, by Lemma 2.1, there is no non-negative integer solution.

Case 3: if $x \geq 1$ and $y \geq 1$, then we consider two cases:

Case 3.1 x is even. We get $4^x \equiv 1 \pmod 5$. Since $n \equiv 1 \pmod 5$, thus $n^y \equiv 1 \pmod 5$. Therefore $z^2 = 4^x + n^y \equiv 2 \pmod 5$ but $\left(\frac{2}{5}\right) = -1$.

Case 3.2 x is odd. We get $4^x \equiv 1 \pmod 3$. Since $n \equiv 1 \pmod 3$, thus $n^y \equiv 1 \pmod 3$. Therefore $z^2 = 4^x + n^y \equiv 2 \pmod 3$ but $\left(\frac{2}{3}\right) = -1$.

Corollary 3.2. The Diophantine equation $4^x + 136^y = z^2$ has no non-negative integer solution x and z.

Proof: Since $136 \equiv 1 \pmod{15}$, by Theorem 3.1 the Diophantine equation $4^x + 136^y = z^2$ has no non-negative integer solution.

Corollary 3.3. Let n be a positive integer where $n \equiv 1 \pmod{15}$. The Diophantine equation $4^x + n^y = u^{2t+6}$ has no non-negative integer solution x, y and u.

Proof: Let $z = u^{t+3}$ then $4^x + n^y = u^{2t+6} = z^2$, $n \equiv 1 \pmod{15}$, which has no solution by Theorem 3.1.

4. Conclusion

In this paper, we discussed the Diophantine equation $4^x + n^y = z^2$ with $n \equiv 1 \pmod{15}$ and x, y, z are non-negative integers. We used the quadratic residue of n which conclusion that Diophantine equation $4^x + n^y = z^2$ has no non-negative integer solution x, y and z.

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