On the Diophantine Equation $4^x + n^y = z^2$

Wachirarak Orosram$^1$, Sasikarn Niratsrok$^2$ and Arisa Sukkharin$^3$

$^{1,2,3}$Department of Mathematics, Faculty of Science, Buriram Rajabhat University
Muang Buriram 31000, Thailand.

$^1$E-mail: wachirarak.tc@bru.ac.th; $^2$E-mail: 620112210028@bru.ac.th
$^3$Corresponding author. $^4$E-mail: 620112210040@bru.ac.th

Received 12 October 2022; accepted 8 December 2022

Abstract. Let $n$ be a positive integer where $n \equiv 1 \pmod{15}$. In this paper we shown that the Diophantine equation $4^x + n^y = z^2$ has no non-negative integer solution where $x, y$ and $z$ are non-negative integers.

Keywords: Diophantine equation, Quadratic residue, non-negative integer

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

In the past, there was a lot of interest in studying the solution of Diophantine equations. The general form of the Diophantine equation is $a^x + b^y = c^z$ which has been studied in [4]. In 2008, Pumnea and Nicoar [8] studied Diophantine equations of the form $a^x + b^y = z^2$, for example: $2^x + 7^y = z^2$, $2^x + 11^y = z^2$ and $2^x + 13^y = z^2$. Many authors also studied some particular cases of the Diophantine equation $4^x + b^y = z^2$, where $b$ is a fixed number and $b$ is a prime number. In 2011, Suvarnaman, Singta and Chotchaisthit [12] showed that Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no solution in non-negative integers. The following year, Chotchaisthit [3] showed that the Diophantine equation $4^x + p^y = z^2$ has no non-negative integer solution where $x, y$ and $z$ are non-negative integers and $p$ is a prime number. In 2014, Sroysang [11] established that the Diophantine equation $4^x + 10^y = z^2$ has no non-negative integer solution where $x, y$ and $z$ are non-negative integers. In 2016, Srisarakham and Thongmoon [10] solved that the Diophantine equation $48^x + 84^y = z^2$ has a unique non-negative integer solution $(x,y,z) = (1, 0, 7)$. In 2018, Kumar, Gupta and Kishan [5] showed that the Diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$ have no solution where $x, y$ and $z$ are non-negative integers. In the same year, Lu [6] investigated the equation of the form $q^x + p^y = z^2$ with $q$ and $p$ are primes. Particularly, Lu considered the equations $3^x + p^y = z^2$ where $p \equiv 5 \pmod{12}$ and $3^x + b^y = z^2$ where $b \equiv 1 \pmod{4}$ and
p \equiv 5(\text{mod } 12) \text{ or } p \equiv 7(\text{mod } 12). \text{ In the next year, Burshtein [1] established some non-negative solutions for the Diophantine equation } 3^r + p^r = z^2 \text{ where } p \text{ is an odd prime number and } x + y \leq 8. \text{ Later, [2] proved that the equation } 8^r + 9^r = z^2 \text{ has no solution when } x, y \text{ and } z \text{ are positive integers by utilizing the last digits of the powers } 8^r, 9^r. \text{ In 2021, Moonchaisook [7] considered the non-linear Diophantine equation } p^r + (p + 4^r)^r = z^2 \text{ has no solution where } p > 3, p + 4^r \text{ are primes.}

In this paper, we consider the Diophantine equation

\[ 2^4 x y z + = \]

where \( n \) is an odd prime and \( x y + \leq 8 \). Later, [2] proved that the equation

\[ 2^8 9^x y z + = \]

has no solution when \( x, y, z \) are positive integers by utilizing the last digits of the powers \( 8, 9^x \).

2. Preliminaries

Let \( p \) be an odd prime and \( a \) be a positive integer where \( \gcd(a, p) = 1 \). If the quadratic congruence \( x^2 \equiv a (\text{mod } p) \) has a solution, then \( a \) is said to be a quadratic residue of \( p \). Otherwise, \( a \) is called a quadratic non-residue of \( p \). In 1798 Adrien-Marie Legendre [9] introduced the Legendre symbol \( \left( \frac{a}{p} \right) \) which is defined by

\[
\left( \frac{a}{p} \right) = \begin{cases} 
1 & \text{if } a \text{ is a quadratic residue of } p, \\
-1 & \text{if } a \text{ is a quadratic non-residue of } p.
\end{cases}
\]

In this paper, using the following symbols;

**Lemma 2.1.** The Diophantine equation \( 4^r + 1 = z^2 \) has no non-negative integer solution where \( x \) and \( z \) are non-negative integers.

**Proof:** Let \( x \) and \( z \) are non-negative integers. If \( x = 0 \), then \( z^2 = 2 \), which is impossible. If \( x = 1 \), then \( z^2 = 5 \), which is impossible. If \( x > 1 \), then \( 4^r + 1 = z^2 \). Since \( 4^r \equiv 1(\text{mod } 3) \), thus \( z^2 = 4^r + 1 \equiv 2(\text{mod } 3) \) but \( \left( \frac{2}{3} \right) = -1 \), this equation has no solution.

Let \( n \equiv l(\text{mod } 15) \). We get \( 15 | n - 1 \) or \( n - 1 = 15k \) for some integers \( k \). Get \( n = 15k + 1 = 3(5k) + 1 = 5(3k) + 1 \) so \( n \equiv 1(\text{mod } 3) \) and \( n \equiv 1(\text{mod } 5) \). In this paper, we assume that \( n \) is a non-negative integer.

**Lemma 2.2.** Let \( n \) be a positive integer with \( n \equiv 1(\text{mod } 15) \). The Diophantine equation \( 1 + n^y = z^2 \) has no non-negative integer solution \( y \) and \( z \).

**Proof:** Let \( y \) and \( z \) are non-negative integers and \( n \) be a positive integer with \( n \equiv l(\text{mod } 15) \) is clear that \( n \equiv l(\text{mod } 3) \) and \( n \equiv l(\text{mod } 5) \). We divide it into two cases as follows:
On the Diophantine Equation \(4^x + n^y = z^2\)

**Case 1:** if \(y = 0\), then \(2 = z^2\) is impossible.

**Case 2:** if \(y \geq 1\), then \(1 + n^y = z^2\). Since \(n \equiv 1 \pmod{5}\), thus \(n^y \equiv 1 \pmod{5}\) and \(z^2 = 1 + n^y \equiv 2 \pmod{5}\) but \(\left(\frac{2}{5}\right) = -1\).

### 3. Main theorem

**Theorem 3.1.** Let \(n\) be a positive integer where \(n \equiv 1 \pmod{15}\). The Diophantine equation \(4^x + n^y = z^2\) has no non-negative integer solution \(x, y\) and \(z\).

**Proof:** Let \(n\) be a positive integer where \(n \equiv 1 \pmod{15}\), and \(x, y, z\) are non-negative integers. We divide it into three cases as follows:

- **Case 1:** \(x = 0\), by Lemma 2.2, there is no non-negative integer solution.
- **Case 2:** \(y = 0\), by Lemma 2.1, there is no non-negative integer solution.
- **Case 3:** if \(x \geq 1\) and \(y \geq 1\), then we consider two cases:

  **Case 3.1** \(x\) is even. We get \(4^x \equiv 1 \pmod{15}\). Since \(n \equiv 1 \pmod{5}\), thus \(n^y \equiv 1 \pmod{5}\) and \(z^2 = 4^x + n^y \equiv 2 \pmod{5}\) but \(\left(\frac{2}{5}\right) = -1\).

  **Case 3.2** \(x\) is odd. We get \(4^x \equiv 1 \pmod{3}\). Since \(n \equiv 1 \pmod{3}\), thus \(n^y \equiv 1 \pmod{3}\) and \(z^2 = 4^x + n^y \equiv 2 \pmod{3}\) but \(\left(\frac{2}{3}\right) = -1\).

**Corollary 3.2.** The Diophantine equation \(4^x + 136^y = z^2\) has no non-negative integer solution \(x\) and \(z\).

**Proof:** Since \(136 \equiv 1 \pmod{15}\), by Theorem 3.1 the Diophantine equation \(4^x + 136^y = z^2\) has no non-negative integer solution.

**Corollary 3.3.** Let \(n\) be a positive integer where \(n \equiv 1 \pmod{15}\). The Diophantine equation \(4^x + n^y = u^{2x+6}\) has no non-negative integer solution \(x, y\) and \(u\).

**Proof:** Let \(z = u^x\) then \(4^x + n^y = u^{2x+6} = z^2\), \(n \equiv 1 \pmod{15}\), which has no solution by Theorem 3.1.

### 4. Conclusion

In this paper, we discussed the Diophantine equation \(4^x + n^y = z^2\) with \(n \equiv 1 \pmod{15}\) and \(x, y, z\) are non-negative integers. We used the quadratic residue of \(n\) which conclusion that Diophantine equation \(4^x + n^y = z^2\) has no non-negative integer solution \(x, y\) and \(z\).

**Acknowledgement.** The authors thank the reviewer for putting valuable remarks and comments on this paper.
Conflict of interest. The authors declare that they have no conflict of interest.

Authors’ Contributions. All the authors contributed equally to this work.

REFERENCES